TSINGHUA-PRINCETON-COMBUSTION INSTITUTE

2025 SUMMER SCHOOL ON COMBUSTION

# Dynamics of Flames and Detonations in Premixed Gas

Paul Clavin

Aix-Marseille Université, France

July 06-12, 2025

# TSINGHUA-PRINCETON-COMBUSTION INSTITUTE

# 2025 SUMMER SCHOOL ON COMBUSTION

**Key Activities / 重要活动**

| | | | |
|---|---|---|---|
| July 6<br>(Sunday)<br>/7月6日<br>(周日) | 10:00-17:30 | Registration<br>注册 | Northeast Gate, Lee Shau Kee Sci. and Tech. Building<br>李兆基科技大楼东北门 |
| | 18:00 | Welcome Reception<br>开班仪式 | A-278, Multifunction Room, Lee Shau Kee Sci. and Tech. Building<br>李兆基科技大楼多功能厅 |

**Class Schedule / 课程安排**

| | | | | | |
|---|---|---|---|---|---|
| July 7-11<br>(Monday-Friday)<br>/7月7-11日<br>(周一至周五) | Morning<br>上午 | 9:00-9:50<br>10:00-10:50<br>11:00-11:50 | Combustion Chemistry<br>Lecturer: Philippe Dagaut<br>**Jianhua Building**<br>**建华楼A109** | Turbulent Combustion<br>Lecturer: Hong G. Im<br>**Jianhua Building**<br>**建华楼LG1-21** | |
| | Afternoon<br>下午 | 14:00-14:50<br>15:00-15:50<br>16:00-16:50 | Dynamics of Flames and Detonations in Premixed Gas<br>Lecturer: Paul Clavin<br>**Jianhua Building**<br>**建华楼A109** | Advanced Laser Diagnostics for Chemically Reacting Flows<br>Lecturer: Mark Linne<br>**Jianhua Building**<br>**建华楼A404** | Applications of Combustion Science to Fire Safety<br>Lecturer: José L. Torero<br>**Jianhua Building**<br>**建华楼LG1-11** |

**Special Activities / 特殊活动**

| | | | |
|---|---|---|---|
| July 6<br>(Sunday)<br>/7月6日<br>(周日) | 13:30-17:30 | Art Museum Visit / 艺术博物馆参观 | Tsinghua University Art Museum<br>清华大学艺术博物馆 |
| July 7<br>(Monday)<br>/7月7日<br>(周一) | 17:00-17:30 | Group Picture Taking / 暑期学校合影 | The open-air plaza next to the New Tsinghua Auditorium<br>天大广场(新清华学堂露天广场) |
| July 8<br>(Tuesday)<br>/7月8日<br>(周二) | 17:00-18:00 | Campus Tour / 校园游览 | Tsinghua University<br>清华大学 |
| July 9<br>(Wednesday)<br>/7月9日<br>(周三) | 18:30-19:30<br>19:30-21:00 | Poster Presentation / 海报展示<br>Career Panel / 职业发展论坛 | B-518, Lee Shau Kee Sci. and Tech. Building<br>李兆基科技大楼B-518会议室 |
| July 10<br>(Thursday)<br>/7月10日<br>(周四) | 18:00 | Farewell Reception / 欢送会 | Guan Chou Yuan Restaurant<br>观畴园餐厅 |
| July 11<br>(Friday)<br>/7月11日<br>(周五) | 8:00-18:00 | Program Certificate Distribution / 学习证书发放 | Jianhua Building<br>建华楼 |
| July 12<br>(Saturday)<br>/7月12日<br>(周六) | 9:30-11:30 | CCE Laboratory Tour / 燃烧能源中心实验室参观 | Northeast Gate, Lee Shau Kee Sci. and Tech. Building<br>李兆基科技大楼东北门 |

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

**Paul Clavin**

Aix-Marseille Université
**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

*Theoretical analyses + laboratory experiments*


Aix-en-Provence
Marseille
Jan Mayen
(NORWAY)
Norwegian
Sea
NORWAY
ICELAND
Reykjavík
Faroe
Islands
(DEN.)
Tórshavn
SWEDEN
FINLAND
Murmansk
White Sea
Arkhangel'sk
Gulf
of
Bothnia
Lake
Ladoga
Lake
Onega
Helsinki
Oslo
Stockholm
Tallinn
EST.
Saint Petersburg
Rockall
(U.K.)
North
Sea
DENMARK
Copenhagen
Baltic
Sea
Riga
LAT.
RUSSIA
LITH.
Vilnius
Yaroslavl'
Nizhniy
Novgorod
Moscow
Kazan'
Ul'yanovsk
Glasgow
UNITED
KINGDOM
Belfast
Leeds
Dublin
IRELAND
Isle of
Man
(U.K.)
Birmingham
London
Minsk
BELARUS
Hamburg
Berlin
Warsaw
Łódź
POLAND
Amsterdam
NETH.
Brussels
Lille
BELGIUM
LUX.
Luxembourg
Cologne
GERMANY
Frankfurt
Homyel'
Voronezh
Saratov
Kyiv
L'viv
UKRAINE
Kharkiv
Prague
CZECH REP.
Kraków
SLOVAKIA
Bratislava
Vienna
AUSTRIA
Munich
LIECH.
Guernsey (U.K.)
Jersey (U.K.)
Paris
FRANCE
Volgograd
Donets'k
MOLDOVA
Chişinău
Budapest
HUNGARY
SLOVENIA
Ljubljana
Bern
SWITZ.
Lyon
Milan
Turin
Zagreb
CROATIA
Odesa
Rostov
Astrakhan'
Sea of
Azov
ROMANIA
Belgrade
Bucharest
SERBIA
BOS. &
HER.
Sarajevo
SAN
MARINO
Genoa
MONACO
Bordeaux
Toulouse
Bay of
Biscay
Bilbao
ANDORRA
ITALY
Black Sea
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(FR.)
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CITY
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Porto
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(SP.)
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(IT.)
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(IT.)
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Oran
Melilla
(SP.)
Tunis
TUNISIA
MALTA
Valletta
Adana
Gaziantep
Mosul
Erbil
Aleppo
SYRIA
Nicosia
CYPRUS
Beirut
LEBANON
Damascus
IRAQ
Baghdad
Crete
(GR.)
Mediterranean Sea
MADEIRA
ISLANDS
(PORT.)
Rabat
Casablanca
Fès
Tripoli
Tel Aviv-Yafo
ISRAEL
Jerusalem
Amman
JORDAN
Marrakech
MOROCCO
Benghāzī
Alexandria
Cairo
Al Başra

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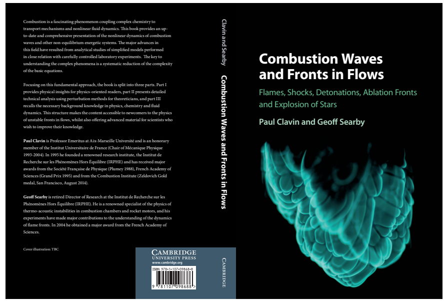



**Combustion Waves and Fronts in Flows**
*P. Clavin and G. Searby*
*Cambridge University Press 2016*

# Lecture 1: Orders of magnitude

1-1: Overall combustion chemistry

1-2: Combustion waves in gaseous mixtures

1-3: Arrhenius law

1-4: Hydrocarbon/air flames

1-5: Instabilities of flames

# 1-1: Overall combustion chemistry

*Lavoisier 1777*

$$\text{reactants} \longrightarrow \text{products} + \text{heat release}$$

binding energy of small molecules $\approx$ a few eV

$1eV/\text{molecule} \approx 23$ kcal/mole $\quad \Rightarrow \quad T_b - T_u \approx 2000$ K

$(C_p \approx 10\,\text{cal/mole/K})$

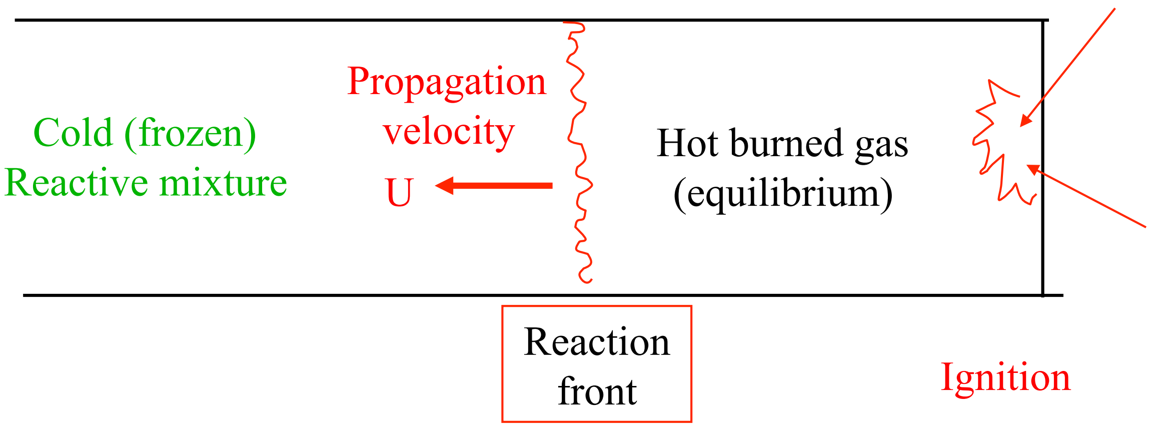
reaction time $\tau_r(T)$ extremely sensitive to temperature:

$T < 500K : \tau_r(T) \approx \infty$ (frozen mixture of reactants)

$T \approx 2500K : \tau_r(T) \approx 10^{-6}s.$

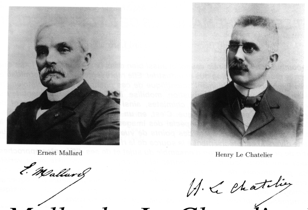
*Euler 1738*

thermal feedback $\Rightarrow$ combustion waves




*Davy 1830*

# 1-2: Combustion waves in gaseous mixtures

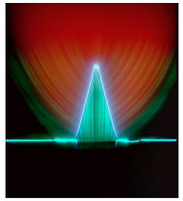


*Mallard, Le Chatelier 1883*

acethylen/oxygen

Flames : $10\ \mathrm{cm/s} - 10\ \mathrm{m/s}, \quad \Delta p/p \approx -10^{-5}$

Laminar propagation



*Laser Tomography L.Boyer 1980*

*Bec Bunsen J. Quinard 2000*

Fast deflagrations : $\approx 100\ \mathrm{m/s}, \quad \Delta p/p \approx -10^{-1}$

Turbulent propagation

*John H.S. Lee 1990*

*Berthelot, Vielle 1884*

Detonations : $\approx 2000\ \mathrm{m/s}, \quad \Delta p/p \approx +30$

3000 m/s

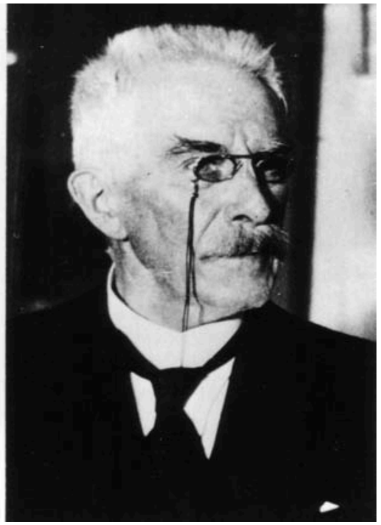
Cellular structure


*Shchelkin 1960*

Marcelin Berthelot (1827-1907)

Paul Vieille (1854-1934)

Ernest Mallard (1833-1899)

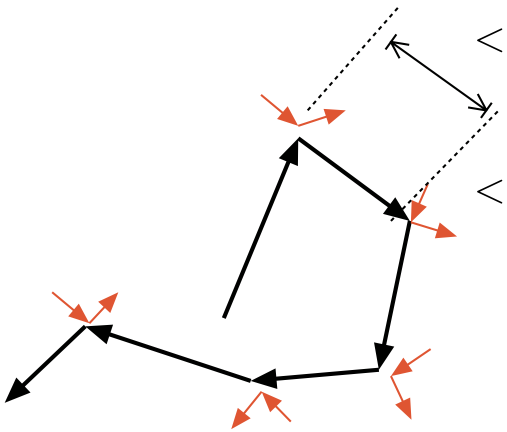


Henry Le Châtelier (1850-1936)

Yakov Borisovich Zeldovich (1914-1987)

## Back to the kinetic theory of gases

### Binary collisions of molecules



$< \text{distance} >= l$

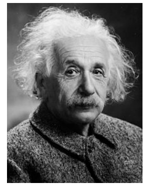
mean free path

$< \text{velocity} >= V \equiv \sqrt{3k_BT/m}$

$< \text{time} >= \tau_{coll} \equiv l/V$

Maxwell 1867

Bolzmann 1877



Einstein 1905

speed of sound: $a \approx V$

$\tau_{coll}$ : collision time $\approx$ relaxation time

## Equilibrium state. The Maxwell-Boltzmann distribution

$m$ : mass of molecules, $n$ : number density, $T$ : temperature, $k_B$ : Boltzmann cst.

$$f^{(eq)}(\mathbf{v}, n, T)\,\mathrm{d}^3\mathbf{v}\,\mathrm{d}^3\mathbf{r} = n\frac{m^{3/2}}{(2\pi k_BT)^{3/2}}\mathrm{e}^{-m|\mathbf{v}|^2/(2k_BT)}\mathrm{d}^3\mathbf{v}\,\mathrm{d}^3\mathbf{r}$$

velocity of molecules

element of volume

## Molecular diffusion $\equiv$ Random Walk

spreading : $\dfrac{1}{(4\pi Dt)^{3/2}}\mathrm{e}^{-r^2/4Dt}$

$D = lV = l^2/\tau_{coll} \approx a^2\tau_{coll}$

Diffusion coefficient

# Parameters

chemical energy/unit mass $q_m \quad \Rightarrow \quad T_b/T_u = 5-10$

sound speed, $\quad a_b/a_u = \sqrt{T_b/T_u}$

reaction rate $1/\tau_r(T_b) \quad \Rightarrow 1/\tau_r(T_b) \approx 3\times 10^5 \mathrm{s}^{-1}$

molecular and thermal diffusion coefficients $D \approx D_T$

$(\text{length})^2/\text{time}$

**Detonation= shock driven reaction wave; Flame=reaction-diffusion wave**

# Dimensional analysis

chemical energy/mass

$[q_m] = (\text{velocity})^2$

$[D] = (\text{velocity})^2 \times \text{time}$

$10^{-5}\mathrm{m}^2/\mathrm{s}$

propagation velocity

detonation: $\quad \mathcal{D} \approx \sqrt{q_m} \approx a_b$

$\approx 1000 \text{ m/s}$

supersonic $\quad \mathcal{D}/a_u > 1$

laminar flames: $\quad U_L \approx \sqrt{D/\tau_r(T_b)}$

$\approx 1 \text{ m/s}$

subsonic $\quad U_L/a_u < 1$

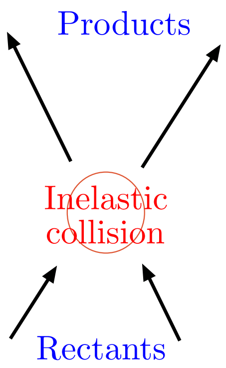
# 1-3: Arrhenius law

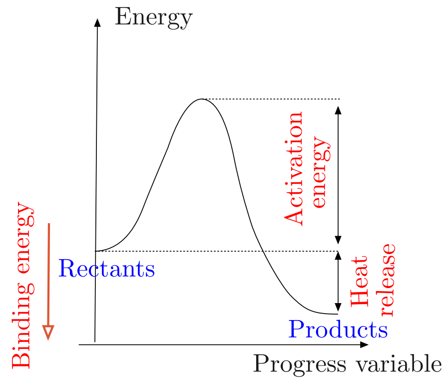
## Overall reaction rate: highly sensitive to temperature, Arrhenius law

large activation energy

$$\boxed{\frac{E}{k_B T_b} \approx 8}$$

Collision in gases




$$e^{-E/k_B T_b} \approx 3 \times 10^{-4}$$

$$T_b/T_u = 8 \;\Rightarrow\; e^{-E/k_B T_u} \approx 1.6 \times 10^{-28}$$

Maxwell-Boltzmann distribution

Kinetic theory of gases $\Rightarrow$ Arrhenius law

high thermal sensitivity

MB distribution $\propto e^{-\frac{1}{2}\frac{mv^2}{k_B T}}$ $\quad\Rightarrow\quad$ $\boxed{\dfrac{1}{\tau_r(T)} = \dfrac{1}{\tau_{coll}} e^{-E/k_B T}}$

Arrhenius factor

elastic collision rate $1/\tau_{coll} \approx 10^9 \mathrm{s}^{-1}$

$\Rightarrow 1/\tau_r(T_b) \approx 3 \times 10^5 \mathrm{s}^{-1}$

$T_b/T_u = 8 \Rightarrow \tau_r(T_u) \approx 10^{10}$ years !!

Kinetic theory of gases $\Rightarrow$ Flame properties

sound speed $\quad$ men free path

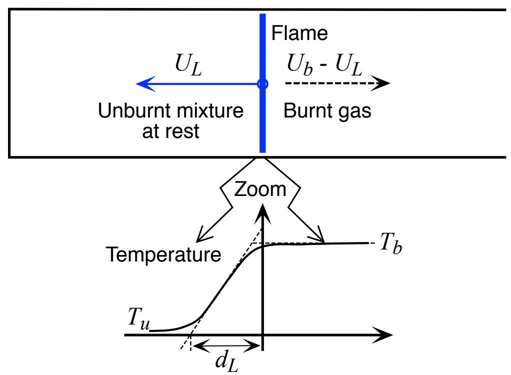
$D \approx a^2\, \tau_{coll} \approx l^2/\tau_{coll}$ $\qquad a \approx l/\tau_{coll}$

laminar flame velocity $\qquad$ sound speed

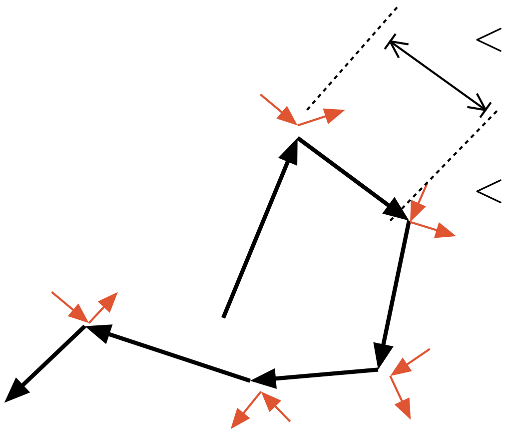
$U_L \approx \sqrt{D_T/\tau_r}$ $\qquad$ subsonic $\qquad U_L/a \approx \sqrt{e^{-E/k_B T_b}} \ll 1$

$d_L \approx D_T/U_L$ $\qquad$ macroscopic structure $\qquad d_L \approx l\sqrt{e^{E/k_B T_b}} \gg l$

flame thickness $\qquad$ mean free path



# Molecular diffusion $\equiv$ Random Walk



$<$ distance $>= l$ mean free path

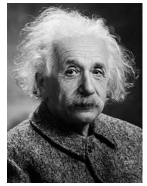
$<$ velocity $>= V \equiv \sqrt{3k_BT/m}$

$a \approx V$

$<$ time $>= \tau_{coll} \equiv l/V$

Maxwell 1867

Bolzmann 1877



Einstein 1905

spreading : $\dfrac{1}{(4\pi Dt)^{3/2}} \mathrm{e}^{-r^2/4Dt}$

$$D = lV = l^2/\tau_{coll} \approx a^2 \tau_{coll}$$

# Flame structure

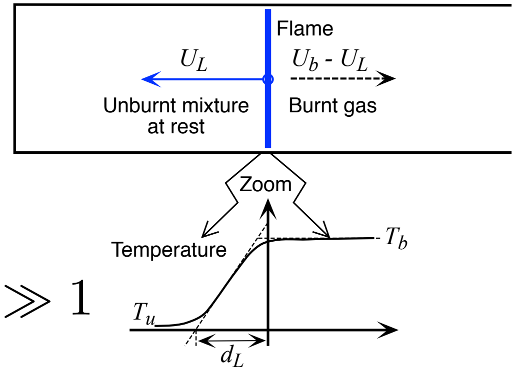
$$U_L \approx \sqrt{D/\tau_r(T_b)}$$

$$\frac{1}{\tau_r(T_b)} = \frac{1}{\tau_{coll}} \mathrm{e}^{-E/k_BT_b}$$

Arrhenius factor

$$U_L/a \approx \sqrt{\mathrm{e}^{-E/k_BT_b}} \ll 1$$

$$d_L/l \approx U_L\tau_r/l \approx \sqrt{D\tau_r}/l \approx \sqrt{\mathrm{e}^{E/k_BT_b}} \gg 1$$



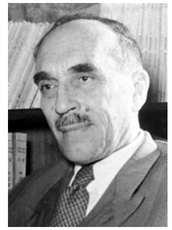

# Limitations of the dimensional analysis

$$\left.\begin{array}{l} \mathrm{e}^{-E/k_BT_b} \approx 3\times 10^{-4} \\ a \approx 500\mathrm{m/s} \\ l \approx 10^{-7}\ \mathrm{m} \end{array}\right\} \Rightarrow \begin{array}{ll} U_L \approx 8.6\,\mathrm{m/s} & \text{too large} \\ d_L \approx 0.6\times 10^{-5}\ \mathrm{m} & \text{too small} \end{array}$$

hydrocarbon/air: $10 - 50$ cm/s, $1 - 10^{-1}$ mm

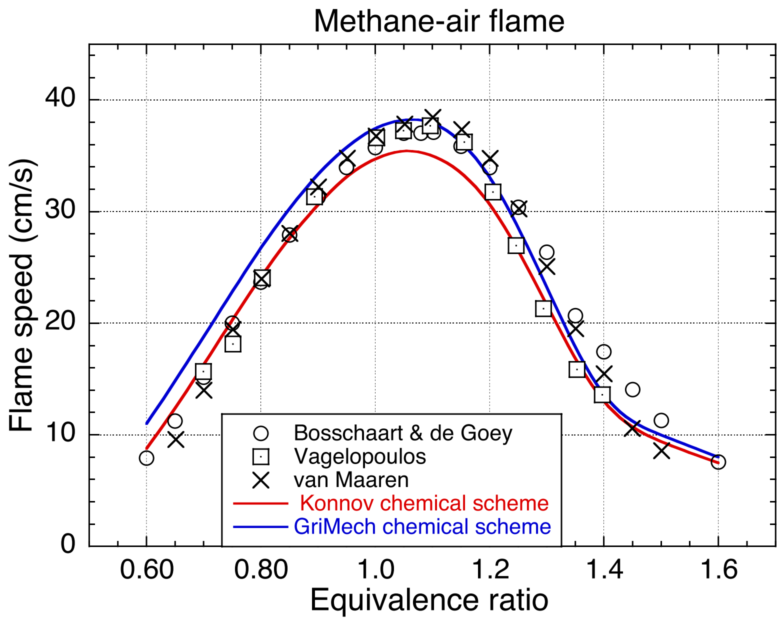
# 1-4: Hydrocarbon/air flames

*Chain reactions in combustion*



*Semenov 1934*

Methane-air flame




$$\vartheta_F^+ F + \vartheta_o^+ O_2 \rightleftharpoons P$$

Equivalence ratio

$$\phi = \frac{N_F / N_{O_2}}{\vartheta_F^+ / \vartheta_{O_2}^+}$$

$\phi = 1$ : stoichiometry

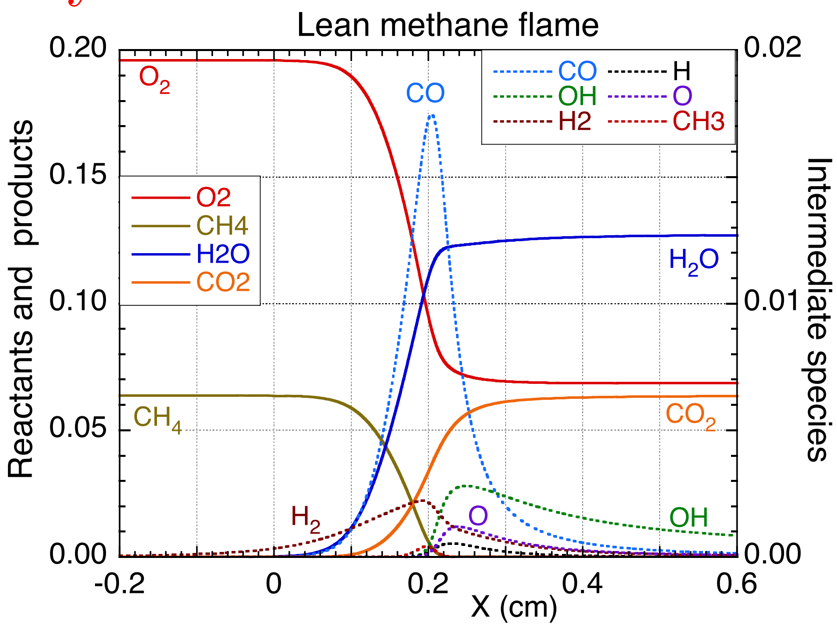
$\phi > 1$ : fuel rich

$\phi < 1$ : fuel lean

near to the flammability limit

$$\phi = 0.65$$

"thicker flame"

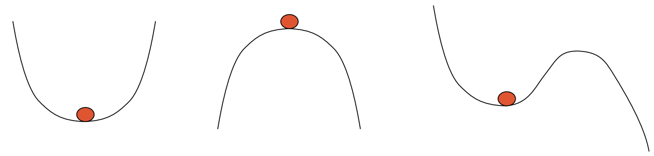



$$d_L \approx U_L \tau_r(T_b)$$
$$\approx \sqrt{D_T \tau_r(T_b)}$$
$$\approx D_T \sqrt{\tau_r(T_b)/D_T}$$
$$\approx D_T / U_L$$

$U_L \searrow \quad d_L \nearrow$

# 1-5: Instabilities of flames

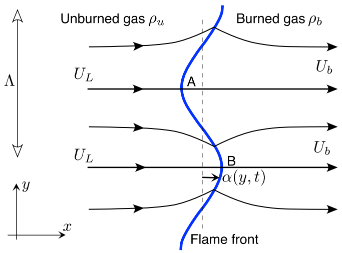
## Intrinsic instabilities



Stable    Linearly unstable    Linearly stable Nonlinearly unstable

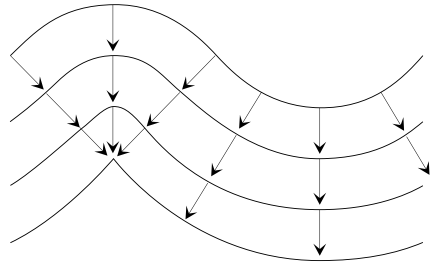
Planar flames are linearly unstable:

- hydrodynamic instability of the flame front $\boxed{\rho_u > \rho_b}$



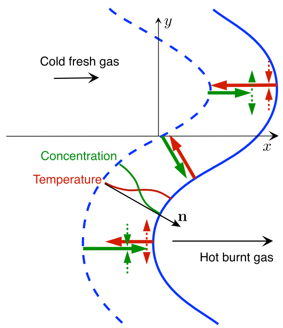

induced flow

Propane lean flame

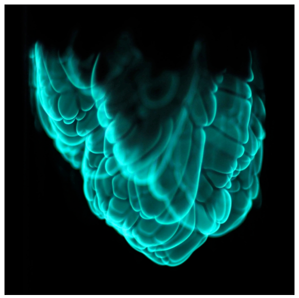


Cusp formation
Huygens construction

- thermo-diffusive instability $\boxed{D_T < D}$



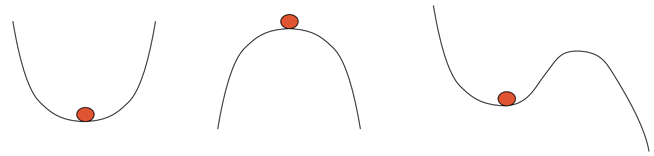

Unstable inner structure



Propane rich flame

# 1-5: Instabilities of flames

## Intrinsic instabilities



Stable | Linearly unstable | Linearly stable Nonlinearly unstable

Planar flames are linearly unstable:

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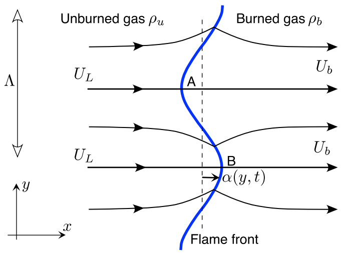
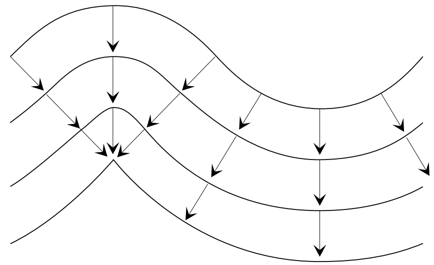


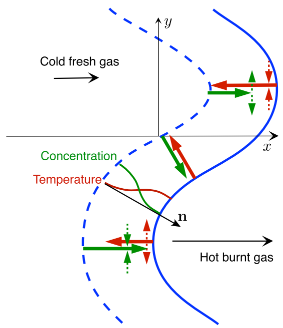

induced flow

Propane lean flame

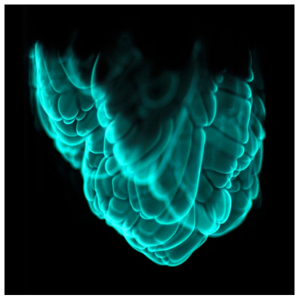


Cusp formation
Huygens construction

- thermo-diffusive instability $\boxed{D_T < D}$




Unstable inner structure



Propane rich flame

# System instability (combustion in a cavity)

The coupling of flames with acoustics can be unstable

Thermo-acoustic instabilities (Rayleigh criterion)

Lord Rayleigh 1878

Combustion chambers

Rocket engine

Gas turbines

# Vibratory instability of flames in tubes

Lean methane-air flame

$$\phi = 0.73 \qquad U_L = 23\ \mathrm{cm/s}$$

$$\phi = 0.8 \qquad U_L = 30\ \mathrm{cm/s}$$


G. Searby IRPHE 2006

G. Searby (1992) Combust. Sci. Technol. 81 pp 221-231

Tomography cut: L.Boyer (1980) Combust. Flame **39** pp 321-323

# Effect of geometrical parameters on thermo-acoustic instability of downward propagating flames in tubes

Ajit Kumar Dubey, Yoichiro Koyama, Nozomu Hashimoto, Osamu Fujita

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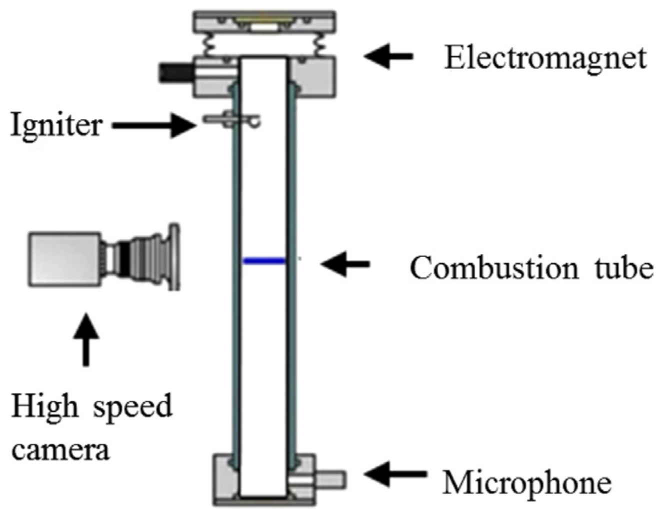
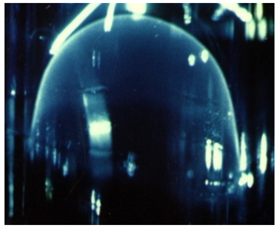
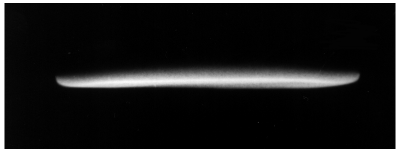


# Effect of acceleration

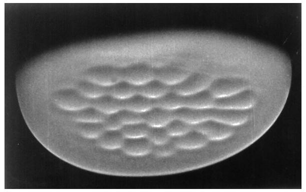
## Gravity



Propane flame propagating upwards

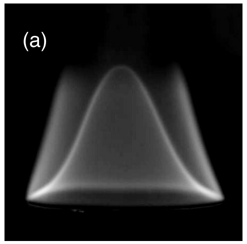


Slow downwards propagating flame

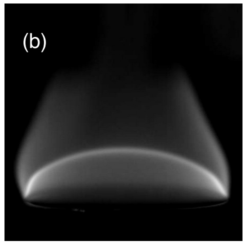


slightly faster

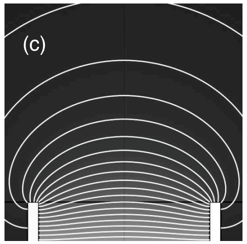
## Effect of an acoustic field on a Bunsen flame




Methane rich Bunsen flame
$\phi = 1.5$




in the presence of
an axial acoustic field




equipotential surface
in the absence of flame

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases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ECM & CNRS


## Lecture 11
## Governing equations of reacting flows

Gaseous mixtures in normal conditions $\approx$ continuum medium in local equilibrium

mean free path $\ll$ macroscopic length

many microscopic particles $\in$ a fluid "particle"

relaxation time towards equilibrium of fluid particles $\ll$ macroscopic time scale

internal structure of shock waves

# Lecture 2: Governing equations

(simplified form, see de Groot et Mazur (1962) or Williams (1985) for more details)

2-1. Conserved extensive quantities

2-2. Continuity

2-3. Fick's law. Diffusion equation

2-4. Conservation of momentum

2-5. Conservation of total energy

*Thermal equation*

*Inviscid flows in reactive gases*

*Conservative forms*

*One-dimensional inviscid and compressible flow*

2.6. Entropy production

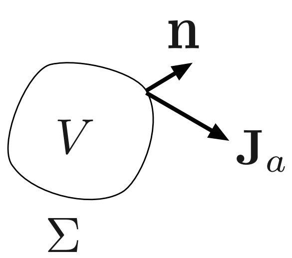
# 2-1. Conserved extensive quantities

extensive quantities $A_V = \iiint\limits_V \rho a \, \mathrm{d}^3\mathbf{r}$ — constant volume

mass weighted distribution $a(\mathbf{r},t)$

mass density $\rho(\mathbf{r},t)$

conservation equation $V$ fixed $\quad \mathrm{d}A/\mathrm{d}t = \iiint\limits_V [\partial(\rho a)/\partial t] \, \mathrm{d}^3\mathbf{r} = (\mathrm{d}A/\mathrm{d}t)_1 + (\mathrm{d}A/\mathrm{d}t)_2$




$$(\mathrm{d}A/\mathrm{d}t)_1 = -\iint\limits_\Sigma \mathbf{n}.\mathbf{J}_a \mathrm{d}^2\sigma \qquad (\mathrm{d}A/\mathrm{d}t)_2 = \iiint\limits_V \dot{\omega}_a(\mathbf{r},t) \, \mathrm{d}^3\mathbf{r}$$

constant surface

$$(\mathrm{d}A/\mathrm{d}t)_1 = -\iiint\limits_V \nabla.\mathbf{J}_a \, \mathrm{d}^3\mathbf{r}$$ Gauss-Ostrogradsky theorem

vector field $\mathbf{J}_a(\mathbf{r},t)$

$$\boxed{\partial(\rho a)/\partial t = -\nabla.\mathbf{J}_a + \dot{\omega}_a}$$

conserved quantities : no production terms $\quad \dot{\omega}_a = 0 \quad$ No volumetric production
(mass, momentum and energy)

conserved scalar $\quad a(\mathbf{r},t) \qquad \boxed{\partial(\rho a)/\partial t = -\nabla.\mathbf{J}_a}$

conserved vector $\quad \mathbf{a}(\mathbf{r},t) \qquad \partial(\rho \mathbf{a})/\partial t = -\nabla.\underline{\underline{\mathbf{J}}}_a$

tensor field $\quad \underline{\underline{\mathbf{J}}}_a(\mathbf{r},t)$

## 2-2. Continuity

mass is a conserved scalar (classical mechanics)

$$\partial\rho/\partial t = -\nabla.\mathbf{J} \qquad \mathbf{J} \equiv \rho\mathbf{u} \qquad \partial\rho/\partial t = -\nabla.(\rho\mathbf{u})$$

$$\nabla.(\rho\mathbf{u}) = \mathbf{u}.\nabla\rho + \rho\nabla.\mathbf{u}$$

material (convective) derivative $\dfrac{\mathrm{D}}{\mathrm{D}t} \equiv \dfrac{\partial}{\partial t} + \mathbf{u}.\nabla$ $\qquad \dfrac{1}{\rho}\dfrac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}$

continuity equation

$v \equiv 1/\rho$ $\qquad \dfrac{1}{v}\dfrac{\mathrm{D}v}{\mathrm{D}t} = \nabla.\mathbf{u}$

volume/unity of mass

---

# Lagrangian form of conservation equation

convection flux

$$\partial(\rho a)/\partial t = -\nabla.\mathbf{J}_a + \dot{\omega}_a \qquad \boxed{\rho\mathrm{D}a/\mathrm{D}t = -\nabla.\mathbf{J}'_a + \dot{\omega}_a} \qquad \boxed{\mathbf{J}_a \equiv \rho a\mathbf{u} + \mathbf{J}'_a}$$

conserved scalar:

diffusion flux ?

$$\partial(\rho a)/\partial t = -\nabla.\mathbf{J}_a \qquad \boxed{\rho\mathrm{D}a/\mathrm{D}t = -\nabla.\mathbf{J}'_a}$$

(definition of the diffusion flux in the equation for energy is slightly different)
see slide 11

# 2-3. Fick's law. Diffusion equation

mass fraction $\quad Y_i = \rho_i/\rho \quad \sum_i Y_i = 1$

inert mixture $\quad$ mass fraction of species is a conserved scalar

$$\rho \mathrm{D}Y_i/\mathrm{D}t = -\nabla.\mathbf{J}'_i \qquad \sum_i \mathbf{J}'_i = 0$$

Kinetic theory of gas (binary diffusion in an abundant species)

Fick's law : $\boxed{\mathbf{J}'_i = -\rho D_i \nabla Y_i}$ $\qquad \rho \mathrm{D}Y_i/\mathrm{D}t = \nabla.[\rho D_i \nabla Y_i]$

$D_i > 0$

diffusion equation

$\rho D_i \approx$ cst. $\qquad \mathbf{u} = 0 \qquad \boxed{\partial Y_i/\partial t = D_i \Delta Y_i}$

archetype of irreversible phenomenon

(random walk)

# Diffusive damping. Dissipative phenomenon

Diffusion eq. (linear) $\partial Y/\partial t = D\Delta Y$ $\quad D > 0 \quad$ $[D] = (length)^2/time$

## Fourier anaysis

$$Y(\mathbf{r},t) = \sum_{\mathbf{k}} \tilde{Y}_{\mathbf{k}}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{r}} \qquad k = |\mathbf{k}| \text{ wave vector}$$

$$\mathrm{d}\tilde{Y}_{\mathbf{k}}(t)/\mathrm{d}t = -(Dk^2)\tilde{Y}_{\mathbf{k}}(t) \qquad \tilde{Y}_{\mathbf{k}}(t) = \tilde{Y}_{\mathbf{k}}(0)\mathrm{e}^{-Dk^2t}$$

Fourier 1824

## Green function Self-similar solution

$$\partial G/\partial t = D\Delta G$$

$$t = 0 : \ G(\mathbf{r},0) = \delta(\mathbf{r}) \qquad G(\mathbf{r},t) = \frac{1}{(4\pi Dt)^{3/2}}\mathrm{e}^{-r^2/4Dt} \qquad r = |\mathbf{r}| \qquad \iiint G(\mathbf{r},t)\mathrm{d}^3\mathbf{r} = 1$$

probability distribution of the test particle

expanding Gaussian: mean thickness $\bar{d} = \sqrt{Dt}$

## number density

$$\partial n/\partial t = D\Delta n$$

$$n(\mathbf{r},t) = NG(\mathbf{r},t) \qquad n(\mathbf{r},t) = \iiint n(\mathbf{r}',0)G(\mathbf{r}-\mathbf{r}',t)\mathrm{d}^3\mathbf{r}'$$

# 2-4. Conservation of momentum

Momentum is a conserved vector (isolated system)

$$\rho \mathrm{D}\mathbf{u}/\mathrm{D}t = -\nabla.\underline{\underline{\Pi}} \quad -\rho g \mathbf{e}_z,$$

surface force (stress tensor) $\underline{\underline{\Pi}} = p\underline{\underline{\mathbf{I}}} + \underline{\underline{\pi}}$ gravity (body force)

thermodynamic pressure (isotropic)

Viscous stress tensor

$$\underline{\underline{\pi}} \equiv -2\eta(\underline{\underline{\nabla \mathbf{u}}})^{(s)} - \underline{\underline{\mathbf{I}}}(\xi - 2\eta/3)\nabla.\mathbf{u}$$

Navier Stokes equations

gravity

$$\rho \mathrm{D}\mathbf{u}/\mathrm{D}t = -\nabla(p + \rho g z) + \eta \Delta \mathbf{u} + (\xi + \eta/3)\nabla(\nabla.\mathbf{u})$$

Viscous shear diffusivity $D_{vis} = \eta/\rho$

Euler equations

$$\rho \mathrm{D}\mathbf{u}/\mathrm{D}t = -\nabla p$$

non dissipative equations

# 2-5. Conservation of total energy

total energy per unit mass $e_{tot} = |\mathbf{u}|^2/2 + e_T + e_{chem} + \ldots$ $\qquad \delta e_T = c_V \delta T$

*internal energy*

internal energy, $\qquad e_{int} = e_T + e_{chem}$

(Additive in a gas when interactions are neglected)

thermal energy, chemical energy (chemical bonds),

(kinetic + rotational & vibrational energy)

total energy is a conserved scalar

$$\partial(\rho e_{tot})/\partial t = -\nabla.\mathbf{J}_{e_{tot}} \qquad \rho\, \mathrm{D}e_{tot}/\mathrm{D}t = -\nabla.\mathbf{J}'_{e_{tot}} \qquad \mathbf{J}'_{e_{tot}} \equiv \mathbf{J}_{e_{tot}} - \rho e_{tot}\mathbf{u}$$

Question: what is the expression of $\mathbf{J}_{e_{tot}}$?

---

$$\rho \mathrm{D}\mathbf{u}/\mathrm{D}t = -\nabla p$$

**Inviscid flow**

Euler equation $\Rightarrow \quad \dfrac{1}{2}\rho\dfrac{\mathrm{D}}{\mathrm{D}t}|\mathbf{u}|^2 = -\mathbf{u}.\nabla p = -\nabla.(p\mathbf{u}) + p\nabla.\mathbf{u}$

---

**Inert flow** (**1st law** of thermodynamics)

without internal viscous dissipation

internal energy = thermal energy

$$\boxed{\rho \mathrm{D}e_T/\mathrm{D}t = \underbrace{-\nabla.\mathbf{J}'_q}_{\text{heat}} \underbrace{- p\nabla.\mathrm{u}}_{\text{work done}}}$$

$p\nabla.\mathbf{u} = \rho p(\mathrm{D}\rho^{-1}/\mathrm{D}t)$

mass conservation

heat flux,

## Fourier law

(simplest form of heat flux)

$$\boxed{\mathbf{J}'_q = -\lambda\nabla T,}$$

thermal conductivity

## Fourier equation

inert material $e_{chem} = \text{cst.}$

no flow $\mathbf{u} = 0$

$$\boxed{\partial T/\partial t = D_T \Delta T}$$

$\delta e_T = c_V \delta T \qquad$ thermal diffusivity $\quad D_T \equiv \lambda/\rho c_V \qquad [D_T] = (length)^2/time$

Question: what is the expression of $\mathbf{J}_{e_{tot}}$ for a single component inert and inviscid flow ?

*Internal energy* *Heat flux* *Compression*

*1st law:* $\rho \mathrm{D} e_T/\mathrm{D}t = \nabla(\lambda \nabla T) - p\nabla.\mathrm{u}$ *Momentum* $(1/2)\rho \mathrm{D}|\mathbf{u}|^2/\mathrm{D}t = -\nabla.(p\mathbf{u}) + p\nabla.\mathbf{u}$

$$\rho \mathrm{D}\left[e_T + |\mathbf{u}|^2/2\right]/\mathrm{D}t = \nabla(\lambda\nabla T) - \nabla.(p\mathbf{u})$$

$$\partial(\rho e_{tot})/\partial t = -\nabla.\mathbf{J}_{e_{tot}} \qquad \mathbf{J}_{e_{tot}} = \mathbf{J}'_{e_{tot}} + \rho e_{tot}\mathbf{u} \qquad \rho\, \mathrm{D}e_{tot}/\mathrm{D}t = -\nabla.\mathbf{J}'_{e_{tot}}$$

$$\mathbf{J}'_{e_{tot}} = \underbrace{-\lambda\nabla T}_{\textit{diffusion}} + \underbrace{p\mathbf{u}}_{\textit{compression induced energy flux}}$$

---

## Thermal balance of an inert and inviscid flow

$\delta e_T = c_v \delta T$ $\qquad c_v \approx \text{cst.}$ (for simplicity, can be easily removed)

$$\rho c_v \mathrm{D}T/\mathrm{D}t = \nabla.(\lambda\nabla T) - p\nabla.\mathbf{u}$$

$$\text{continuity} \;\Rightarrow\; -p\nabla.\mathbf{u} = \frac{p}{\rho}\frac{\mathrm{D}}{\mathrm{D}t}\rho = \frac{\mathrm{D}}{\mathrm{D}t}p - \rho\frac{\mathrm{D}}{\mathrm{D}t}[(c_p - c_v)T]$$

ideal gas law $p = (c_p - c_v)\rho T.$

$$\boxed{\rho c_p \mathrm{D}T/\mathrm{D}t = \mathrm{D}p/\mathrm{d}t + \nabla.(\lambda\nabla T)}$$

compression conduction

# Reactive flows

elementary reaction

$$\vartheta_1^+ A_1 + \ldots + \vartheta_n^+ A_n \quad \rightleftharpoons \quad \vartheta_1^- A_1 + \ldots + \vartheta_n^- A_n + Q.$$

reaction rate
nb/(volume $\times$ time)

$$\dot{W}^{(j)} \equiv (J_+^{(j)} - J_-^{(j)}) \quad \text{and} \quad \vartheta_i^{(j)} \equiv (\vartheta_i^{-(j)} - \vartheta_i^{+(j)}),$$

stoichiometric coefficient

conservation equation for the species

*Fick law*

$$\mathbf{J}'_i = -\rho D_i \nabla Y_i$$

$$\rho \mathrm{D}Y_i/\mathrm{D}t = -\nabla.\mathbf{J}'_i + \sum_j \vartheta_i^{(j)} \mathsf{m}_i \dot{W}^{(j)},$$

$$\boxed{\rho \frac{\mathrm{D}Y_i}{\mathrm{D}t} = \nabla.(\rho D_i \nabla Y_i) + \underbrace{\sum_j}_{\text{sum over the reactions}} \vartheta_i^{(j)} \mathsf{m}_i \dot{W}^{(j)}(T, p, ..Y_k..)}$$

equation for the chemical energy

$$e_{chem} \equiv \sum_i h_i Y_i$$

$$Q^{(j)} = \overbrace{\underbrace{\sum_{i=1}^{n}}_{\text{sum over the species}} (\vartheta_i^{(j)+} - \vartheta_i^{(j)-}) \mathsf{m}_i h_i(T_o)}^{\text{heat of the j th reaction}},$$

enthalpy of formation per unit of mass of species $i$

$$\boxed{\rho \mathrm{D}e_{chem}/\mathrm{D}t = -\sum_i \nabla.(h_i \mathbf{J}'_i) - \sum_j Q^{(j)} \dot{W}^{(j)}}$$

$$e_{chem} \equiv \sum_i h_i Y_i \qquad \rho \mathrm{D} e_{chem}/\mathrm{Dt} = -\sum_i h_i \nabla.\mathbf{J}'_i - \sum_j Q^{(j)}\dot{W}^j \qquad \mathbf{J}'_i = -\rho D_i \nabla Y_i$$

heat released by the $j^{th}$ reaction

rate of the $j^{th}$ reaction
(number per unit time and unit volume)

---

## 1st law

without internal viscous dissipation

$$e_{int} = e_T + e_{chem} \qquad \rho \mathrm{D} e_{int}/\mathrm{Dt} = -\nabla.\mathbf{J}'_q - p\nabla.\mathrm{u}$$

work done by volume change

$$-\nabla.\mathbf{J}'_q = \nabla.(\lambda \nabla T) - \sum_i h_i \nabla.\mathbf{J}'_i$$

diffusive flux of internal energy

contribution of the species to the diffusive flux of internal energy

$$\rho \mathrm{D}(e_T + e_{chem})/\mathrm{Dt} = \nabla.(\lambda\nabla T) - \sum_i h_i \nabla.J'_i - p\nabla.\mathrm{u}$$

---

## Thermal balance of inviscid flow of reactive gas

$$\delta e_T = c_v \delta T \qquad c_v \approx \text{cst.} \quad \text{(for simplicity, can be easily removed)}$$

$$\rho \mathrm{D}(e_T + e_{chem})/\mathrm{Dt} = \rho c_v \mathrm{D}T/\mathrm{Dt} - \sum_i h_i \nabla.\mathrm{J}'_i - \sum_j Q^{(j)}\dot{W}^{(j)}$$

$$\rho c_v \mathrm{D}T/\mathrm{Dt} - \cancel{\sum_i h_i \nabla.\mathrm{J}'_i} - \sum_j Q^{(j)}\dot{W}^{(j)} = \nabla.(\lambda\nabla T) - \cancel{\sum_i h_i \nabla.\mathrm{J}'_i} - p\nabla.\mathrm{u}$$

$$\rho c_v \mathrm{D}T/\mathrm{Dt} = \nabla.(\lambda\nabla T) - p\nabla.\mathbf{u} + \sum_j Q^{(j)}\dot{W}^{(j)}$$

$$\text{continuity} \Rightarrow \quad -p\nabla.\mathbf{u} = \frac{p}{\rho}\frac{\mathrm{D}}{\mathrm{Dt}}\rho = \frac{\mathrm{D}}{\mathrm{Dt}}p - \rho\frac{\mathrm{D}}{\mathrm{Dt}}[(c_p - c_v)T] \quad \Leftarrow \quad p/\rho = (c_p - c_v)T$$

$$\boxed{\rho c_p \mathrm{D}T/\mathrm{Dt} = \mathrm{D}p/\mathrm{Dt} + \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}}$$

compression conduction chemistry

# Governing equations for inviscid flows of reactive gas

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p, \qquad p = (c_p - c_v)\rho T,$$

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)},$$

$$\rho\frac{\mathrm{D}Y_i}{\mathrm{D}t} = \nabla.(\rho D_i \nabla Y_i) + \sum_j \vartheta_i^{(j)}\mathrm{m}_i\dot{W}^{(j)}.$$

stoichiometric coefficient

# Conservative form of the energy equation (inviscid approximation)

$$e_{tot} \equiv e_{int} + |\mathbf{u}|^2/2, \qquad e_{int} = e_T + e_{chem}$$

$$\rho\mathrm{D}(e_T + e_{chem})/\mathrm{D}t = \nabla.(\lambda\nabla T) - \sum_i h_i \nabla.\mathbf{J}'_i - p\nabla.\mathrm{u} \qquad (1/2)\rho\mathrm{D}|\mathbf{u}|^2/\mathrm{D}t = -\nabla.(p\mathbf{u}) + p\nabla.\mathbf{u}$$

$$\rho\mathrm{D}e_{tot}/\mathrm{D}t = -\nabla.\left[\mathbf{J}_q + \sum_i h_i\mathbf{J}'_i + p\mathbf{u}\right] \qquad \mathbf{J}_q = -\lambda\nabla T \qquad \Rightarrow \qquad \boxed{\mathbf{J}'_{e_{tot}} = \mathbf{J}_q + \sum_i h_i\mathbf{J}'_i + p\mathbf{u}}$$

Lagragian form

$$\partial(\rho e_{tot})/\partial t = -\nabla.[\rho\mathbf{u}e_{tot} + \mathbf{u}p + \mathbf{J}_q + \sum_i h_i\mathbf{J}'_i] = \boxed{-\nabla.[\rho\mathbf{u}(e_{tot} + p/\rho) + \mathbf{J}_q + \sum_i h_i\mathbf{J}'_i]}$$

Eulerian form

convective flux of enthalpy

diffusive flux of total energy

progress variable

$$\rho q_m \mathrm{D}\psi/\mathrm{D}t \equiv \sum_j Q^{(j)}\dot{W}^{(j)}, \qquad \psi \in [0,1]$$

heat released per unit mass

$$\boxed{e_{tot} + p/\rho = c_p T + |\mathbf{u}|^2/2 - q_m\psi}$$

# Viscous flow:

$$\underline{\underline{\Pi}} = p\underline{\underline{\mathbf{I}}} + \underline{\underline{\pi}}$$

shear viscosity, bulk viscosity

$$\underline{\underline{\pi}} \equiv -2\eta(\underline{\underline{\nabla \mathbf{u}}})^{(s)} - \underline{\underline{\mathbf{I}}}(\xi - 2\eta/3)\nabla.\mathbf{u}$$

$$\partial(\rho e_{tot})/\partial t = -\nabla.[\rho \mathbf{u}\,(c_p T + |\mathbf{u}|^2/2 - q_m\psi) + \mathbf{J}_q + \mathbf{u}.\underline{\underline{\pi}}]$$

Ideal gas: $\quad p = (c_p - c_v)\rho T$

## Compressible reacting flow in planar geometry (conservative form)

$$\pi = -\mu \partial u/\partial x \qquad \mu \equiv 4\eta/3 + \xi$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x} \qquad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x}\left(p + \rho u^2 - \mu\frac{\partial u}{\partial x}\right)$$

$$\frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u(c_p T + u^2/2 - q_m\psi) - \lambda\frac{\partial T}{\partial x} - \rho D q_m \frac{\partial \Psi}{\partial x} - \mu u \frac{\partial u}{\partial x}\right]$$

---

## Jumps across a planar wave between 2 uniform flows $\psi_b = 1,\ \psi_o = 0$

(inner structure in **steady state** !!)

$$[\rho u]_-^+ = 0 \quad [c_p T + u^2/2 - q_m\psi]_-^+ = 0 \quad [p + \rho u^2]_-^+ = 0$$

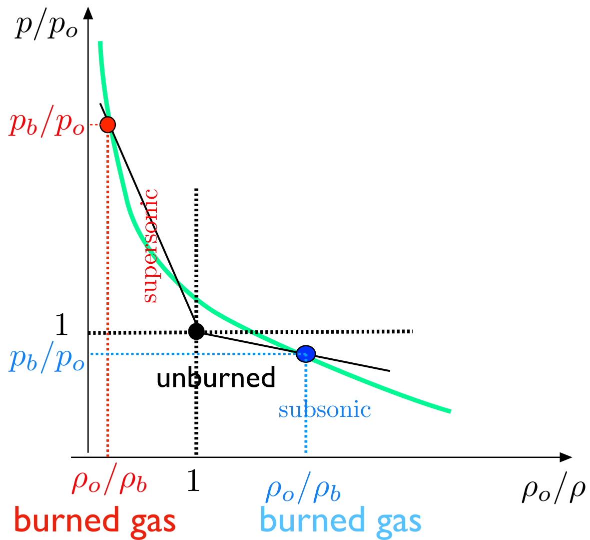
$$u = (\rho_o/\rho)\mathcal{D} \qquad T/T_o = (p/p_o)(\rho_o/\rho)$$

propagation velocity

$$\left(\frac{p}{p_o}, \frac{\rho_o}{\rho}\right) \qquad \left(\frac{p}{p_o} - 1\right) = -\frac{\rho_o \mathcal{D}^2}{p_o}\left(\frac{\rho_o}{\rho} - 1\right)$$

burned gas

burned gas




Rankine (1870) Hugoniot (1889)
$q_m = 0$

Michelson (1893) Rayleigh (1910)
$q_m > 0$

# Compressible and viscous flow in planar geometry

## Conservative form

*Mass* $$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

*Momentum* $$\frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x}\left(p + \rho u^2 - \boxed{\mu \frac{\partial u}{\partial x}}\right)$$

viscous term

*Energy* $$\frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u(c_p T + u^2/2 - q_m \psi) - \lambda \frac{\partial T}{\partial x} - \boxed{\mu u \frac{\partial u}{\partial x}}\right]$$

reacting flow

additional term for a reacting flow

$$-\rho D q_m \frac{\partial \Psi}{\partial x}$$

## Lagrangian Form

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}$$

*Mass* $$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{\partial u}{\partial x}$$

*Momentum* $$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \boxed{\frac{\partial}{\partial x}\left[\mu \frac{\partial u}{\partial x}\right]}$$

Inert flow:

$$e_{tot} = c_v T + u^2/2$$

$$\frac{\partial[\rho(c_v T + u^2/2)]}{\partial t} = -\frac{\partial\left[\rho u(c_v T + u^2/2) + up\right]}{\partial x} + \frac{\partial(\lambda \partial T/\partial x)}{\partial x} + \frac{\partial(\mu u \partial u/\partial x)}{\partial x}$$

$$\rho \frac{D[(c_v T + u^2/2)]}{Dt} = -\frac{\partial(up)}{\partial x} + \frac{\partial(\lambda \partial T/\partial x)}{\partial x} + \frac{\partial(\mu u \partial u/\partial x)}{\partial x}$$

reaction rate

*Energy: extended 1st law*

$$\boxed{\frac{\mathrm{D}(c_v T)}{\mathrm{D}t} = -p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} + \frac{1}{\rho}\frac{\partial}{\partial x}\left[\lambda \frac{\partial T}{\partial x}\right] + \frac{\mu}{\rho}\left(\frac{\partial u}{\partial x}\right)^2 + q_m \dot{w}}$$

*heating by compression variation of the volume* — *heat flux* — *heating by internal viscous dissipation*

# 2.6. Entropy production

$$s(\rho, T, ..Y_i, ..)$$

entropy is a function of state that is not a conserved quantity

$$\partial(\rho s)/\partial t = -\nabla.\mathbf{J}_s + \dot{w}_s,$$

ideal gas

$$\frac{(s - s_o)}{c_v} = \ln\left(\frac{p/\rho^\gamma}{p_o/\rho_o^\gamma}\right)$$

$2^{nd}$ law of thermodynamics dissipative effects $\Rightarrow$ $\boxed{\dot{w}_s \geq 0}$

$$T\delta s = \delta e_T + p\delta v - \sum_i \mu_i \delta Y_i$$

$$T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\mathrm{D}e_T}{\mathrm{D}t} + p\frac{\mathrm{D}(1/\rho)}{\mathrm{D}t} - \sum_i \mu_i \frac{\mathrm{D}Y_i}{\mathrm{D}t}$$

inert mixture:

$$\dot{w}_s = \mathbf{J}'_q.\nabla\left(\frac{1}{T}\right) - \sum_i \mathbf{J}'_i.\nabla\left(\frac{\mu_i}{T}\right) - \frac{1}{T}\underline{\underline{\pi}} : (\underline{\underline{\nabla \mathbf{u}}})^{(s)}$$

**Simple fluid in planar geometry:**

*(inert flow)*

$$J_s = \rho u s - \frac{\lambda}{T}\frac{\partial T}{\partial x},$$

$$\boxed{\dot{w}_s = \frac{\mu}{T}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{\lambda}{T^2}\left(\frac{\partial T}{\partial x}\right)^2}$$

$$\mu > 0, \qquad \lambda > 0$$

$$\rho\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\partial}{\partial x}\left(\frac{\lambda}{T}\frac{\partial T}{\partial x}\right) + \dot{w}_s$$

$$\rho T\frac{\mathrm{D}s}{\mathrm{D}t} = \frac{\partial}{\partial x}\left(\lambda\frac{\partial}{\partial x}T\right) + \mu\left(\frac{\partial u}{\partial x}\right)^2$$

*heat flux* *internal viscous dissipation*

# Various forms of the governing equations

## **Inviscid** reacting flow

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p, \qquad p = (c_p - c_v)\rho T,$$

$$\mathrm{D}Z/\mathrm{D}t \equiv \partial Z/\partial t + \mathbf{u}.\nabla Z \qquad \rho\frac{D(u^2/2)}{Dt} = -\mathbf{u}.\nabla p \qquad \frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \nabla.\mathbf{u}$$

progress variable

$$\dot{W} \equiv \rho\dot{w} \qquad \rho\frac{\mathrm{D}\psi}{\mathrm{D}t} = \nabla.(\rho D\nabla\psi) + \rho\dot{w} \qquad \psi \in [0,1] \qquad \frac{\mathrm{D}\psi}{\mathrm{D}t} = D\Delta\psi + \dot{w}$$

Lagrangian form of the energy eq.

$$\rho c_p\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \nabla.(\lambda\nabla T) + q_m\rho\dot{w} \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} + \frac{D_T}{T}\Delta T + \frac{q_m}{c_p T}\dot{w}$$

$$\rho\mathrm{D}Z/\mathrm{D}t = \partial(\rho Z)/\partial t + \nabla.(\rho\mathbf{u}Z) \qquad D_T = \lambda/(\rho c_p)$$

Conservative form of the energy eq.

$$\frac{\partial}{\partial t}[\rho(c_v T + u^2/2 - q_m\psi)] = -\nabla.\left[\rho\mathbf{u}\left(c_p T + u^2/2 - q_m\psi\right) - \lambda\nabla T + \rho D q_m\nabla\psi\right]$$

$$e_{tot} = c_v T + u^2/2 - q_m\psi \qquad e_{tot} + p/\rho = c_p T + u^2/2 - q_m\psi$$

1st law + heat release.

$$\rho c_v\frac{DT}{Dt} = -p\nabla.\mathbf{u} + \nabla.(\lambda\nabla T) + \rho q_m\dot{w}$$

$$\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \nabla.\mathbf{u} \quad\Rightarrow\quad \frac{1}{\gamma p}\frac{Dp}{Dt} = -\nabla.\mathbf{u} + \frac{D_T}{T}\Delta T + \frac{q_m}{c_p T}\dot{w}$$

$$\frac{1}{p}\frac{Dp}{Dt} - \frac{\gamma}{\rho}\frac{D\rho}{Dt} = \gamma\frac{D_T}{T}\Delta T + \gamma\frac{q_m}{c_p T}\dot{w}$$

ok

Non dissipative case= isentropic

$$\frac{\mathrm{D}(p/\rho^\gamma)}{\mathrm{D}t} = 0 \qquad \frac{\mathrm{D}s}{\mathrm{D}t} = 0 \qquad \mathrm{d}\mathbf{r}(t)/\mathrm{d}t = \mathbf{u}(\mathbf{r},t) \Rightarrow s(\mathbf{r}(t),t) = \text{ constant}$$

# 1-D COMPRESSIBLE and NON-DISSIPATIVE FLOW

$$(1/\rho)\partial p/\partial x + \partial u/\partial t + u\partial u/\partial x = 0$$

$$\gamma p = \rho a^2$$

sound speed

$\pm$ $a$

$$(a^2/\gamma p)\partial p/\partial x + \partial u/\partial t + u\partial u/\partial x = 0 \quad \text{momentum}$$

$$(1/\gamma p)[\partial p/\partial t + u\partial p/\partial x] + \partial u/\partial x = 0 \quad \text{energy+mass}$$

characteristic equations

$$\frac{1}{\rho a}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right] p \pm \left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right] u = 0$$

acoustics: $\delta p = \rho a \delta u$

$$\frac{1}{p}\mathrm{d}p = \frac{\mathrm{d}\rho}{\rho} + 2\frac{\mathrm{d}a}{a}$$

$$\frac{1}{p}\mathrm{d}p = \frac{\gamma}{\rho}\mathrm{d}\rho \qquad \left(1 - \frac{1}{\gamma}\right)\frac{1}{p}\mathrm{d}p = 2\frac{\mathrm{d}a}{a}$$

isentropic

$$\frac{1}{\rho a}\mathrm{d}p = \frac{2}{\gamma - 1}\mathrm{d}a$$

$$\frac{2}{\gamma - 1}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right] a \pm \left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right] u = 0$$

$$\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]\left[\frac{2}{\gamma - 1}a \pm u\right] = 0$$

## Riemann invariants

$$J_\pm \equiv \left[\frac{2}{\gamma - 1}a \pm u\right]$$

$$\left[\frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}\right] J_+ = 0 \qquad \left[\frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}\right] J_- = 0$$

## Piston set in motion from rest

$$x = X_p(t): \quad u = U_p(t) \equiv \frac{\mathrm{d}X_P}{\mathrm{d}t}$$

$$t < 0: \quad U_P = 0, \quad u = 0, \quad a = a_0$$

simple compression wave: $x = \infty: \quad J_- = \dfrac{2}{\gamma - 1}a_o$

$$J_- = \frac{2}{\gamma - 1}a - u = \frac{2}{\gamma - 1}a_0 \qquad a = \frac{\gamma - 1}{2}u + a_0 \qquad J_+ = \frac{2}{\gamma - 1}a_0 + 2u$$

$$u + a = \frac{\gamma + 1}{2}u + a_0 \qquad \frac{\gamma + 1}{2}J_+ = \frac{\gamma + 1}{2}u + \frac{\gamma + 1}{\gamma - 1}a_0$$

$$\frac{\gamma + 1}{2}\left[\frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}\right] J_+ = 0 \quad \Rightarrow \quad \frac{\partial Z}{\partial t} + Z\frac{\partial Z}{\partial x} = 0 \qquad Z \equiv \frac{\gamma + 1}{2}u + a_0$$

Burgers equation without dissipation

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases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ECM & CNRS


## Lecture III
## Thermal propagation of flames

# Lecture 3: Thermal propagation

3-1. Quasi-isobaric approximation (Low Mach number)

3-2. One-step irreversible reaction

3-3. Unity Lewis number and large activation energy

3-4. Zeldovich & Frank-Kamenetskii asymptotic analysis

*Preheated zone*

*Inner reaction layer*

*Matched asymptotic solution*

3-5. Reaction diffusion waves

*Phase space*

*Selected solution in an unstable medium*

# 3-1. Quasi-isobaric approximation (Low Mach number)

order of magnitude $\rho(\mathbf{u}.\nabla)\mathbf{u} \approx -\nabla p \quad \Rightarrow \quad \delta p \approx \rho u \delta u$

$$p \approx \rho a^2 \quad \Rightarrow \boxed{\delta p/p \approx u^2/a^2 \equiv M^2} \quad \Leftarrow \delta u \approx u$$

slow evolution $\partial/\partial t \approx \mathbf{u}.\nabla \ll a\,|\nabla|$

$+$ very subsonic flow $M^2 \ll 1 \quad \Rightarrow \quad \delta p/p \ll \delta T/T = O(1)$

$$\frac{p}{\rho c_p T} = O(1)$$

$$\left|\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t}\right| \ll \left|\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t}\right| \quad \Rightarrow \quad \boxed{\left|\frac{\mathrm{D}p}{\mathrm{D}t}\right| \ll \left|\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t}\right|}$$

$$\rho c_p \mathrm{D}T/\mathrm{D}t = \cancel{\mathrm{D}p/\mathrm{D}t} + \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}$$

quasi-isobaric flame structure

(in open space)

$$\boxed{\rho T = \rho_o T_o \qquad \begin{aligned} \rho c_p \mathrm{D}T/\mathrm{D}t &= \nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}(T,..Y_k..) \\ \rho \mathrm{D}Y_i/\mathrm{D}t &= \nabla.(\rho D_i \nabla Y_i) + \sum_j \vartheta_i^{(j)}\mathsf{m}_i\dot{W}^{(j)}(T,..Y_k..), \end{aligned}}$$

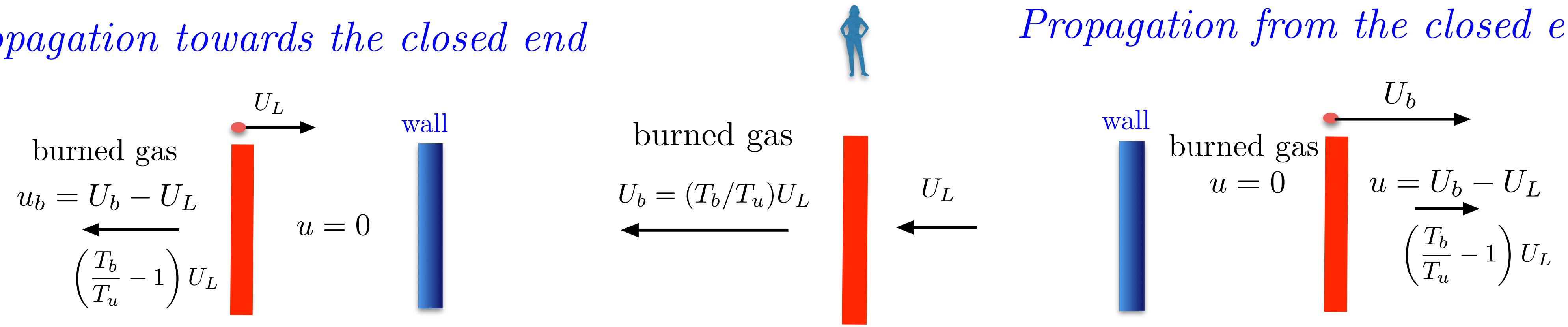
# 1-D flame in a tube

one end is closed and the other open

Quasi-steady inner structure: Velocity of the front / Fresh mixture $U_L = \text{cst.}$

*Propagation towards the closed end*

*Propagation from the closed end*




reference frame of the lab

wall at rest

reference frame of the flame

reference frame of the lab

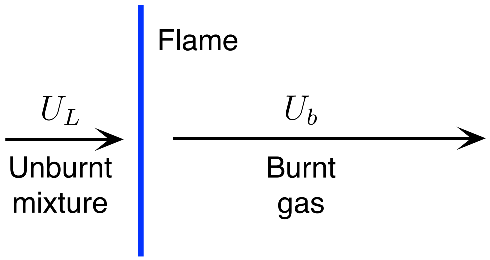
wall at rest

conservation of mass: $\rho U = \text{cst.}$

quasi-isobaric approximation: $\rho T \approx \text{cst.}$

$$\boxed{\frac{\rho_u}{\rho_b} = \frac{T_b}{T_u} = \frac{U_b}{U_u} > 1}$$

# Problem set up of the steady inner structure of a planar flame




reference frame of the flame

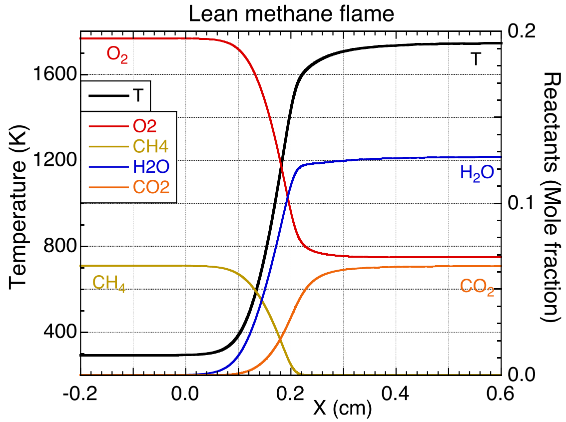
$$\rho\, \mathrm{D}/\mathrm{Dt} = m \mathrm{d}/\mathrm{d}x$$

mass: $\rho u =$ constant

mass flux across the flame

$$m \equiv \rho_u U_L = \rho_b U_b, \qquad U_b/U_L \approx T_b/T_u, \approx 4-8$$

quasi-isobaric approximation: $\rho T \approx$ cst.




axis oriented towards the burned gas

energy : $$m c_p \frac{\mathrm{d}T}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda \frac{\mathrm{d}T}{\mathrm{d}x}\right) = \sum_j Q^{(j)} \dot{W}^{(j)}(T, ..Y_i..)$$

species : $$m \frac{\mathrm{d}Y_i}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho D_i \frac{\mathrm{d}Y_i}{\mathrm{d}x}\right) = \sum_j \vartheta_i^{(j)} M_i \dot{W}^{(j)}(T, ..Y_{i'}..),$$

$m?$

closed system of equations for $T$ and $Y_i$

eigenvalue: flame velocity

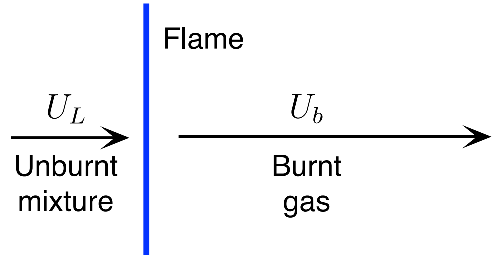
boundary conditions:

$$x = -\infty: \;\; T = T_u, \;\; Y_i = Y_{iu}, \;\; \dot{W}^{(j)} = 0$$ unburned gas frozen state

$$x = +\infty: \;\; \mathrm{d}T/\mathrm{d}x = 0, \;\; Y_i = Y_{ib}, \;\; \dot{W}^{(j)} = 0$$ burned gas equilibrium state

# 3-2. One-step irreversible reaction

$$R \rightarrow P + Q$$




$R$ in an inert ; $Y$ = mass fraction of $R$

## Velocity and structure of the planar flame

(reference frame of the flame) $\rho u = \text{constant}$ (mass flux)

$$mc_p \frac{\mathrm{d}T}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\lambda \frac{\mathrm{d}T}{\mathrm{d}x}\right) = \rho q_R \dot{W}$$

$q_R$ = energy released per unit of mass of R

$m \equiv \rho_u U_L$ unknown

$$m \frac{\mathrm{d}Y}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\rho D \frac{\mathrm{d}Y}{\mathrm{d}x}\right) = -\rho \dot{W}$$

$$mY_u = \int_{-\infty}^{+\infty} \rho \dot{W} \,\mathrm{d}x$$

$x \rightarrow -\infty :$ $\quad Y = Y_u,\ T = T_u$ unburned

$x \rightarrow +\infty :$ $\quad Y = 0 \quad \dfrac{\partial T}{\partial x} = 0$ burned

$$\boxed{c_p(T_b - T_u) = q_m \equiv q_R Y_u}$$

## Arrhenius law

$$\rho \dot{W} = \rho_b \frac{Y}{\tau_r(T)} \qquad \frac{1}{\tau_r(T)} \equiv \frac{\mathrm{e}^{-E/k_B T}}{\tau_{coll}} \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

$$\boxed{\frac{1}{\tau_r(T)} = \frac{1}{\tau_{rb}} \mathrm{e}^{-\frac{T_b}{T}\beta(1-\theta)}} \qquad \beta \equiv \frac{E}{k_B T_b}\left(1 - \frac{T_u}{T_b}\right) \qquad \theta \equiv \frac{T - T_u}{T_b - T_u} \in [0,1]$$

Reduced activation energy

Reduced temperature

# 3-3. Unity Lewis number and large activation energy

$\mathrm{Le} \equiv D_T/D$ Reduced temperature and mass fraction

$$\theta \equiv \frac{T-T_u}{T_b-T_u} \in [0,1] \qquad \psi \equiv \frac{Y}{Y_u} \in [0,1] \qquad \beta \equiv \frac{E}{k_B T_b}\left(1-\frac{T_u}{T_b}\right)$$

$$m\frac{\mathrm{d}\theta}{\mathrm{d}x} - \rho D_T \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \rho\frac{\dot W}{Y_{1u}}, \qquad m\frac{\mathrm{d}\psi}{\mathrm{d}x} - \frac{\rho D_T}{\mathrm{Le}}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = -\rho\frac{\dot W}{Y_{1u}}, \qquad \rho\frac{\dot W}{Y_u} = \rho_b\frac{\psi}{\tau_{rb}}\mathrm{e}^{-\frac{T_b}{T}\beta(1-\theta)}$$

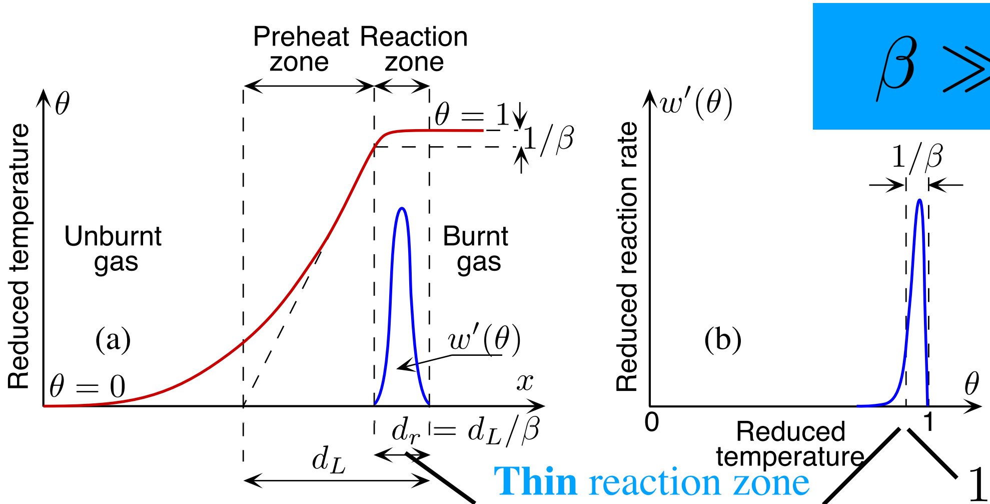
$$x=-\infty\ : \theta=0, \psi=1, \qquad x=+\infty\ : \theta=1, \psi=0 \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

$\mathrm{Le}=1$ $\qquad \psi = 1-\theta$

$$\psi = 1-\theta \qquad m\frac{\mathrm{d}\theta}{\mathrm{d}x} - \rho D_T \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \rho\frac{\dot W}{Y_{1u}}, \qquad \rho\frac{\dot W}{Y_u} = \rho_b\frac{(1-\theta)}{\tau_{rb}}\mathrm{e}^{-\frac{T_b}{T}\beta(1-\theta)}$$

$$x=-\infty\ : \theta=0, \qquad x=+\infty\ : \theta=1,$$

$\beta \gg 1$




Thin reaction zone

$1-\theta = O(1/\beta)$

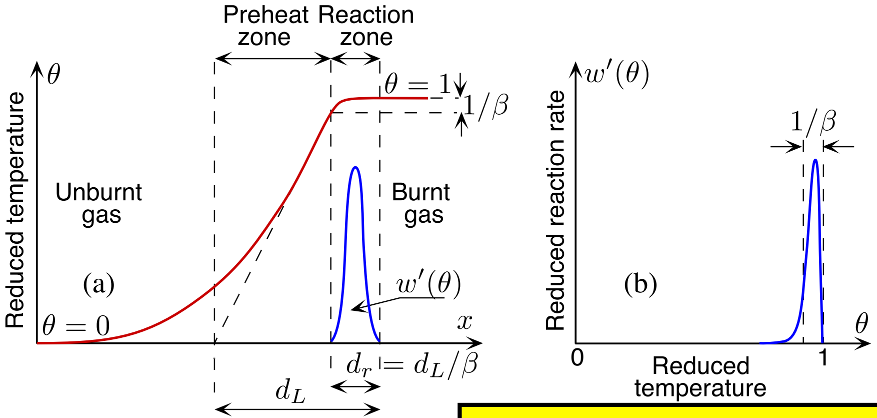
$$\rho\frac{\dot W}{Y_u} = \rho_b\frac{w'(\theta)}{\tau_{rb}}$$

$$w'(\theta) \approx (1-\theta)\mathrm{e}^{-\beta(1-\theta)}$$

(reaction rate is non negligible only when $T \approx T_b$)

# 3-4. Zeldovich & Frank-Kamenetskii asymptotic analysis

Ya. B. Zeldovich, D.A. Frank-Kamenetskii Acta Phys. Chem. (1938) **9**, 341-350




$$\boxed{\beta \longrightarrow \infty}$$

$$\beta \equiv \frac{E}{k_B T_b}\left(1 - \frac{T_u}{T_b}\right)$$

Zeldovich 1938

$$\boxed{m\frac{\mathrm{d}\theta}{\mathrm{d}x} - \rho D_T \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \rho\frac{\dot{W}}{Y_{1u}},}$$

$m?$

$$x = -\infty \;\; : \theta = 0, \qquad x = +\infty \;\; : \theta = 1,$$

$$\boxed{w'(\theta) \approx (1-\theta)\mathrm{e}^{-\beta(1-\theta)}} \qquad \boxed{\rho\dot{W}/Y_u = \rho_b w'(\theta)/\tau_{rb}} \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

## preheated zone $\dot{W} \approx 0$

$$m\mathrm{d}\theta/\mathrm{d}x - \rho D_T \mathrm{d}^2\theta/\mathrm{d}x^2 = 0 \qquad \rho D_T = \text{cst.} \qquad \boxed{\theta = \mathrm{e}^{mx/\rho D_T}}$$

origin $x = 0$ : location of the reaction zone $\theta = 1$

$$d_L \equiv \rho D_T/m = D_{T_u}/U_L$$

## matching condition

heat flux into the preheated zone

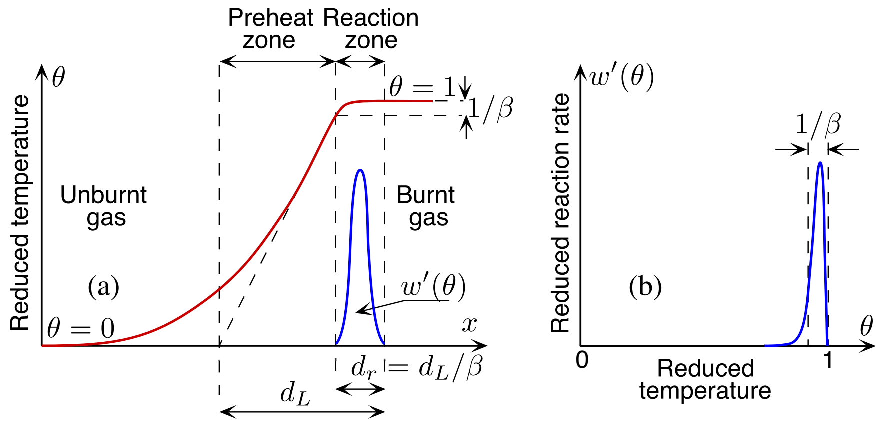
$$\rho D_T \mathrm{d}\theta/\mathrm{d}x|_{\theta=1} = m$$

should be equal to the heat flux from the thin reaction layer

$$m\frac{\mathrm{d}\theta}{\mathrm{d}x} - \rho D_T \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \frac{\rho_b}{\tau_{rb}}(1-\theta)\mathrm{e}^{-\beta(1-\theta)}$$

# Inner reaction layer

$$\boxed{1-\theta = O(1/\beta)} \qquad w'(\theta) \approx (1-\theta)\mathrm{e}^{-\beta(1-\theta)} = O(1/\beta)$$




$$U_L \propto D_T/d_L$$

$$m = \rho D_T/d_L$$

$$\delta\theta = O(1/\beta)$$

$$m\frac{\mathrm{d}\theta}{\mathrm{d}x} \approx \cancel{m\frac{\delta\theta}{d_r} \approx \frac{\rho_b D_{Tb}}{d_r d_L}\frac{1}{\beta}} \Leftarrow$$

$$\rho_b D_{Tb}\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \approx \rho_b D_{Tb}\frac{\delta\theta}{d_r^2} \approx \frac{\rho_b D_{Tb}}{d_r^2}\frac{1}{\beta}$$

$$\frac{\rho_b}{\tau_{rb}}w'(\theta) \approx \frac{\rho_b}{\tau_{rb}}\delta\theta \approx \frac{\rho_b}{\tau_{rb}}\frac{1}{\beta}$$

$$\boxed{d_r \ll d_L} \qquad d_r \approx \sqrt{D_{Tb}\tau_{rb}}$$

$$\cancel{m\mathrm{d}\theta/\mathrm{d}x} - \rho D_T \mathrm{d}^2\theta/\mathrm{d}x^2 = \frac{\rho_b}{\tau_{rb}}(1-\theta)\mathrm{e}^{-\beta(1-\theta)}$$

$$-\frac{D_{Tb}}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \approx \frac{1}{\tau_{rb}}(1-\theta)\mathrm{e}^{-\beta(1-\theta)}\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$\boxed{\Theta = \beta(1-\theta)} \Rightarrow \tau_{rb}\frac{D_{Tb}}{2}\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \approx \int_\theta^1 (1-\theta)\mathrm{e}^{-\beta(1-\theta)}\mathrm{d}\theta = \frac{1}{\beta^2}\int_0^{\beta(1-\theta)} \Theta \mathrm{e}^{-\Theta}\mathrm{d}\Theta$$

# Asymptotic solution $\beta \to \infty$

$$\int_0^\infty \Theta \mathrm{e}^{-\Theta}\mathrm{d}\Theta = 1$$

upstream exit of the inner layer $\quad \beta(1-\theta) \to \infty \;:\; D_{Tb}\mathrm{d}\theta/\mathrm{d}x \to \sqrt{(2/\beta^2)D_{Tb}/\tau_{rb}}$

downstream entrance of the external zone $\quad \theta \to 1 : \; \rho_b D_{Tb}\mathrm{d}\theta/\mathrm{d}x|_{\theta=1} = m$

matching

$$\boxed{m = \rho_b\sqrt{(2/\beta^2)D_{Tb}/\tau_{rb}},} \quad U_L = m/\rho_u, \Rightarrow \boxed{d_r/d_L = O(1/\beta)}$$

$$U_L \approx \sqrt{D_{Tb}/\tau_{rb}}$$

dimensional analysis

# SUMMARY

## Flame=quasi-isobaric reaction-diffusion wave.
## The flame velocity is highly sensitive to temperature

ZFK result for a one-step first order reaction in the limit $\beta \equiv \dfrac{E}{k_B T_b}\left(1 - \dfrac{T_u}{T_b}\right) \gg 1$

$$U_b = \sqrt{\frac{2}{\beta^2}\frac{D_{Tb}}{\tau_{rb}}} \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

$$D_{Tb} \approx a_b^2 \tau_{coll} \qquad \frac{U_b}{a_b} = \frac{1}{\beta}\sqrt{2\,\mathrm{e}^{-E/k_B T_b}} \ll 1$$

$\dfrac{E}{k_B T_b} \gg 1:$ markedly subsonic

high thermal sensitivity $\quad \dfrac{\delta U_b}{U_b} \approx \dfrac{\beta}{2}\dfrac{\delta T_b}{T_b}, \quad \beta \gg 1$

Laminar flame velocity/unburned gas $\quad U_L = (T_u/T_b)U_b \in [15\ \mathrm{cm/s} - 9\ \mathrm{m/s}]$

Transit time $\quad \tau_L = (d_L/U_L) \in [10^{-4}\mathrm{s} - 10^{-3}\mathrm{s}]$

## 3.5) Reaction-diffusion waves with a smooth reaction rate ✕ Flames

non-dimensional form

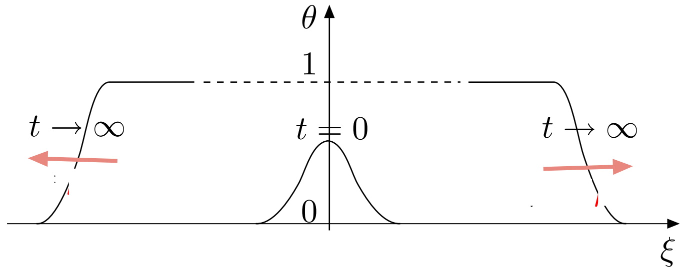
$$\tau \equiv \frac{t}{\tau_r}, \quad \xi \equiv \frac{x}{\sqrt{D\tau_r}}$$

$$\frac{\partial\theta}{\partial\tau} - \frac{\partial^2\theta}{\partial\xi^2} = w(\theta) \qquad \theta \geq 0 \qquad \theta \in [0,1]$$

for example: $\omega(\theta) = \theta(1-\theta)$

propagation of steady state $\theta = 1$ into steady state $\theta = 0$

steady state: $\omega = 0$




stability of steady states

stable: $\mathrm{d}w/\mathrm{d}\theta < 0,$

unstable: $\mathrm{d}w/\mathrm{d}\theta > 0,$

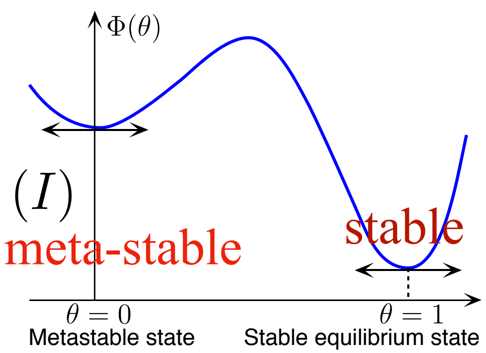
(equilibrium state) $\quad \theta = 1, \quad w = 0$ stable steady state

two different cases depending on the property of the initial state $\theta = 0$

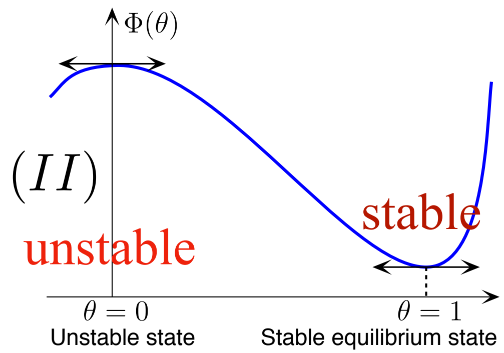
initial state:

$(I): \quad \theta = 0, \quad w = 0$ metastable steady state (less stable)

$(II): \quad \theta = 0, \quad w = 0$ unstable steady state (out of equilibrium)




$$\mathrm{d}\Phi/\mathrm{d}\theta = -w(\theta)$$



# propagating planar wave at constant velocity

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial x^2} = w(\theta)$$ traveling wave solution (from right to left)

$\theta \in [0,1]$ $\qquad \theta(\xi)$

Larangian coordinate: $\quad \xi \equiv x + \mu t$

$$\partial/\partial t = \mu\, \mathrm{d}/\mathrm{d}\xi$$
$$\partial/\partial x = \mathrm{d}/\mathrm{d}\xi$$

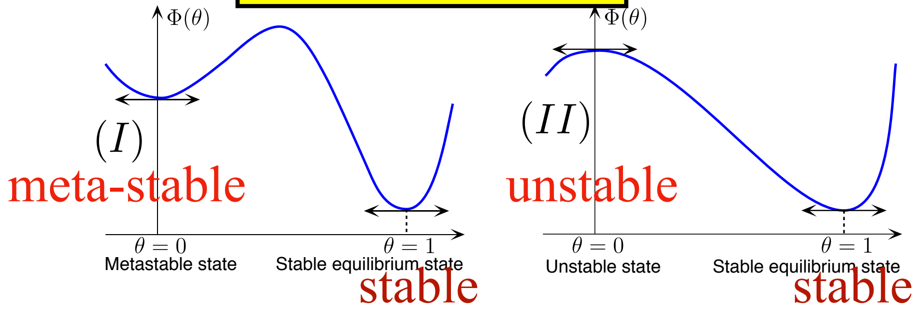
propagation velocity ? $\qquad$ ODE: $$\boxed{\mu \frac{\mathrm{d}\theta}{\mathrm{d}\xi} - \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = w(\theta)}$$

$\mu$ is an eigenvalue of the problem

$$\boxed{\mathrm{d}\Phi/\mathrm{d}\theta = -w(\theta)}$$

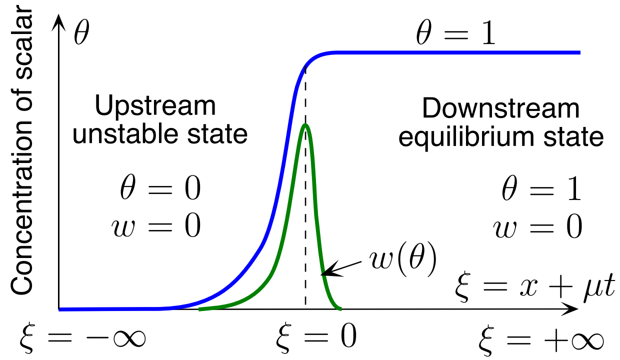
$\xi = -\infty: \quad \theta = 0, \ w = 0,$
initial state

$\xi = +\infty: \quad \theta = 1, \ w = 0$
final (equilibrium) state



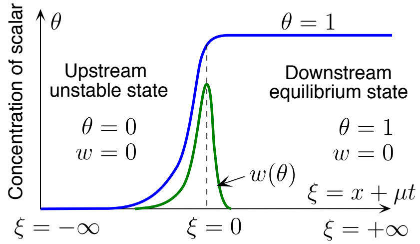

$\mu$ unknown, $\quad$ number of solutions ?

ZFK flame model: $\omega > 0$, case $(II)$



# Number of solutions ? phase space, phase portrait

$$\theta \geqslant 0$$




second order system

$$X \equiv \theta, \qquad Y \equiv \mu \mathrm{d}\theta/\mathrm{d}\xi$$

$$\boxed{\mathrm{d}X/\mathrm{d}\xi = Y/\mu \qquad \mathrm{d}Y/\mathrm{d}\xi = \mu\,[Y-\omega(X)]}$$

$$\mu\frac{\mathrm{d}X}{\mathrm{d}\xi} - \frac{\mathrm{d}^2X}{\mathrm{d}\xi^2} = \omega(X) \qquad \mu\,?$$

$$\boxed{\mathrm{d}Y/\mathrm{d}X = \mu^2[Y-w(X)]/Y}$$

Linearisation about $\boxed{X=0, Y=0 \quad \xi\to-\infty}$ and $\boxed{X=1, Y=0 \quad \omega'_\theta<0 \quad \xi\to\infty \text{ (equilibrium state)}}$

fixed points: $\mathrm{d}X/\mathrm{d}\xi = \mathrm{d}Y/\mathrm{d}\xi = 0$

$Y=0$, $\mathrm{d}Y/\mathrm{d}\xi = 0$

$\mathrm{d}Y/\mathrm{d}X = 0/0$

Two eigenvalues $r_+$ and $r_-$ and two eigendvectors $k_+$ and $k_-$

$$\delta X = A_+e^{\xi\, r_+} + A_-e^{\xi\, r_-}, \qquad \delta Y = k_+A_+e^{\xi\, r_+} + k_-A_-e^{\xi\, r_-}$$

$$\mu\frac{\mathrm{d}\delta X}{\mathrm{d}\xi} - \frac{\mathrm{d}^2\delta X}{\mathrm{d}\xi^2} = \omega'_\theta\delta X \qquad \mu r - r^2 - \omega'_\theta = 0$$

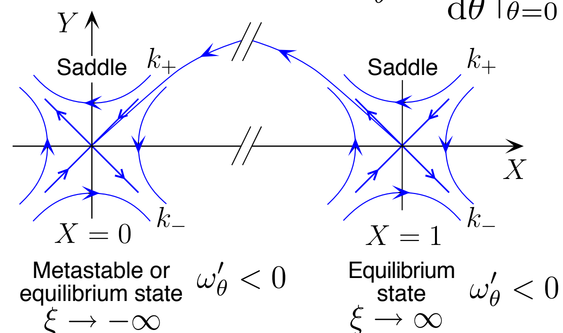
$$Y = \mu\frac{\mathrm{d}X}{\mathrm{d}\xi} \qquad \left[\frac{\mathrm{d}Y}{\mathrm{d}X}\right]_\pm = k_\pm$$

$$\boxed{2r_\pm = \mu \pm \sqrt{\mu^2-4\omega'_\theta} \qquad k_\pm = \mu r_\pm}$$

$$\omega'_\theta \equiv \mathrm{d}\omega/\mathrm{d}\theta|_{\theta=0,1}$$

## case $(I)$

stable initial state $\omega'_\theta \equiv \left.\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right|_{\theta=0} < 0$

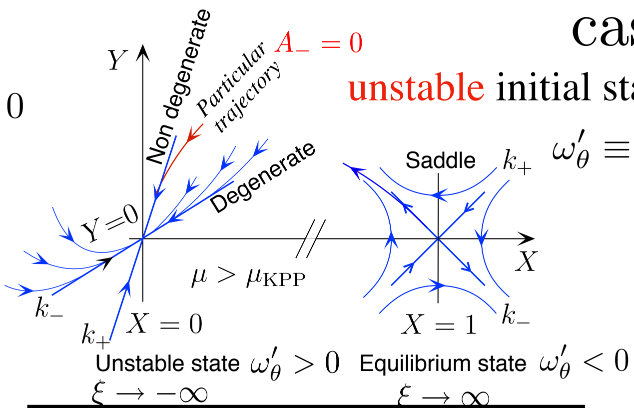



$r_-<0,\ r_+>0$ $\qquad$ $r_+>0,\ r_-<0$

One solution

## case $(II)$

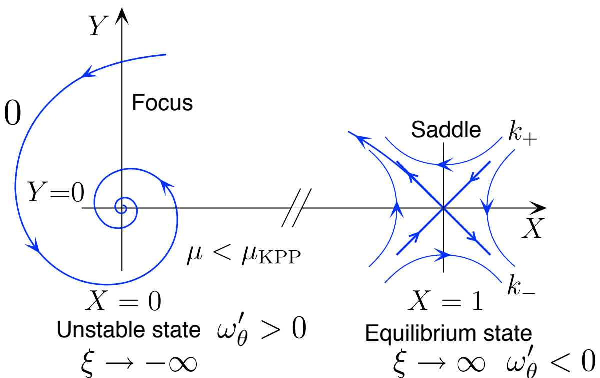
unstable initial state $\omega'_\theta \equiv \left.\frac{\mathrm{d}\omega}{\mathrm{d}\theta}\right|_{\theta=0} > 0$




$r_+>r_->0$ $\qquad$ $r_+>0,\ r_-<0$

Infinite numbers of solutions

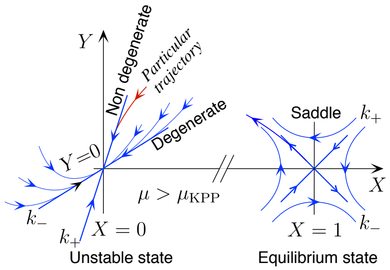
One particular solution

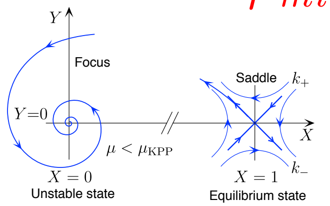



$\mathrm{Im}\, r_- \neq 0,\ \mathrm{Im}\, r_+ \neq 0\ \mathrm{Re}\, r_\pm > 0$ $\qquad$ $r_+>0,\ r_-<0$

No solution $(\theta \geqslant 0)$

# Unstable medium $\left.\frac{d\theta}{dt}\right|_{\theta=0} > 0$




$\mu$ wave velocity

$\mu_{mini}$

Infinite number of travelling wave solutions (continuous set)

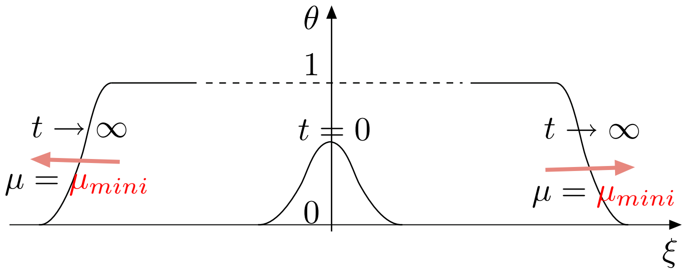
continuous spectrum with a lower bound

$$2r_\pm = \mu \pm \sqrt{\mu^2 - 4\omega'_\theta}$$

Kolmogorov 1937

$\mu_{mini} \equiv 2\sqrt{dw/d\theta|_{\theta=0}}$ lower bound : $r_+ = r_- = \mu/2$ $k_+ = k_-$

soft nonlinear term $\omega(\theta)$ soft case $\Rightarrow$ collapse of the 2 eigenvalues




the lower bound solution is selected $\mu_{mini} \equiv 2\sqrt{dw/d\theta|_{\theta=0}}$ $\neq$ $\boxed{\mu_{\rm flame} \propto \sqrt{2\int_0^1 w(\theta)d\theta}}$

$\partial\theta/\partial t - \partial^2\theta/\partial\xi^2 = \omega'_o\theta,$ where $\omega'_o \equiv \partial w/\partial\theta|_{\theta=0} > 0$

Foot of the temperature profile

$\theta(\xi,t) \equiv Z(\xi,t)e^{+\omega'_o t}$ $\quad \partial Z/\partial t - \partial^2 Z/\partial\xi^2 = 0$

$Z = e^{-\xi^2/4t}/\sqrt{t}$ $\quad \theta \propto \exp[-\xi^2/4t + \omega'_o t - \ln(t)/2]$ $\quad \boxed{\xi^2 \approx 4\omega'_o t^2}$

OK for a soft term $\omega(\theta)$

Wrong for a stiff term $\omega(\theta)$

The lower bound solution changes of nature when $\omega(\theta)$ get stiffer

Clavin, Liñan (1984) in NATO ASI Series B. Physics vol. 116 pp 292-338

Soft $\omega(\theta) = \theta(1-\theta)$

Stiff $w(\theta,\beta) = (\beta^2/2)\theta(1-\theta)e^{-\beta(1-\theta)}, \ \beta \gg 1$

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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ECM & CNRS

## Lecture IV
## Hydrodynamic instability of flames

# Lecture 4 : Hydrodynamic instability of flames

4-1. Jump across an hydrodynamic discontinuity

4-2. Linearized Euler equations of an incompressible fluid

4-3. Conditions at the front

4-4. Dynamics of passive interfaces

4-5. Darrieus-Landau instability

4-6. Curvature effect: a simplified approach

# $IV-1)$ Jump across an hydrodynamic discontinuity

flame considered as a discontinuity
flame thickness and curvature neglected

$$\Lambda \gg d_L$$

$$\rho_b < \rho_u, \qquad \frac{\rho_b}{\rho_b} \approx \frac{1}{8}$$

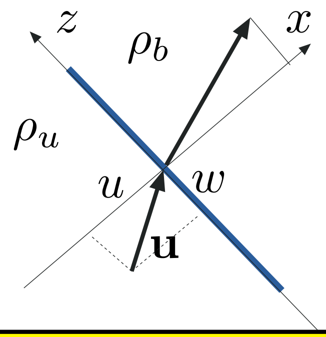
Flame is treated as a surface of discontinuity (zero thickness) separating two incompressible flows

Low Mach nb approx + inviscid approx: Euler eqs

$$\partial\rho/\partial t = -\nabla.(\rho\mathbf{u}) \qquad \rho\mathrm{D}\mathbf{u}/\mathrm{D}t = -\nabla p \;\Leftrightarrow\; \partial(\rho\mathbf{u})/\partial t = -\nabla.(p\underline{\underline{\mathbf{I}}} + \rho\underline{\underline{\mathbf{uu}}})$$

tilted planar front

reference frame of the flame $\qquad \mathbf{r} = (x, z), \quad \mathbf{u} = (u, w)$




2 dim

$$\partial\rho/\partial t = -\partial(\rho u)/\partial x - \partial(\cancel{\rho w})/\partial z,$$

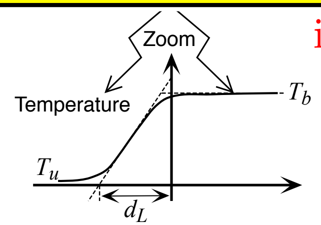
$$\partial(\rho u)/\partial t = -\partial(p + \rho u^2)/\partial x - \partial(\cancel{\rho uw})/\partial z,$$

$$\partial(\rho w)/\partial t = -\partial(\rho uw)/\partial x - \partial(\cancel{p + \rho w^2})/\partial z$$

$$\lim_{d_L \to 0} \int_{d_L} a(x, y, t)\mathrm{d}x = 0$$

if $a(\mathbf{r}, t)$ is regular

jumps in the normal direction (reference frame of the flame)



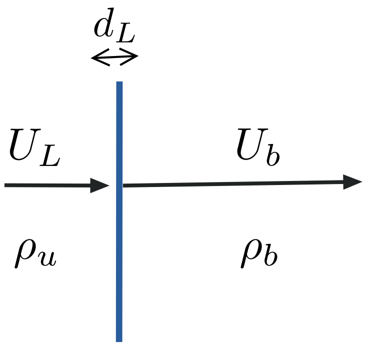

$$[\rho u]_-^+ = 0 \qquad \rho u = \rho_u U_L = \rho_b U_b$$

$$[p + \rho u^2]_-^+ = 0$$

$$\rho u \neq 0 \quad \Rightarrow \quad [w]_-^+ = 0$$

$$\lim_{d_L \to 0} \int_{d_L} \frac{\partial a}{\partial x}\mathrm{d}x = [a]_-^+$$

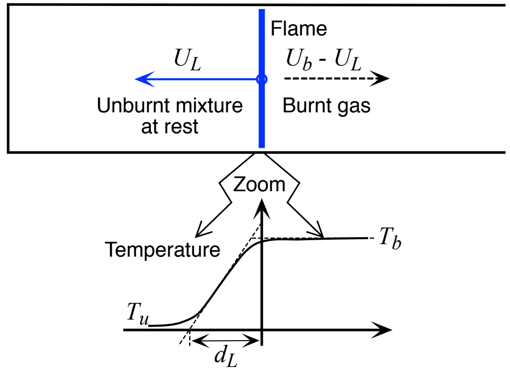
$$\boxed{\rho_u > \rho_b}$$



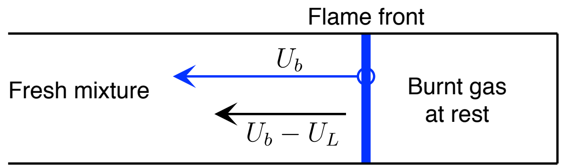

reference frame of the flame front

$$\frac{U_b}{U_L} = \frac{\rho_u}{\rho_b} = \frac{T_b}{T_u}$$

conservation of mass + isobaric approx

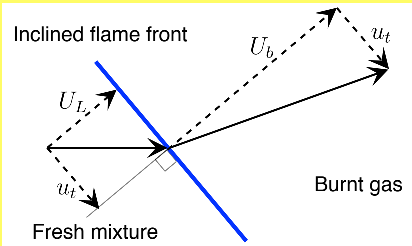



Piston effect

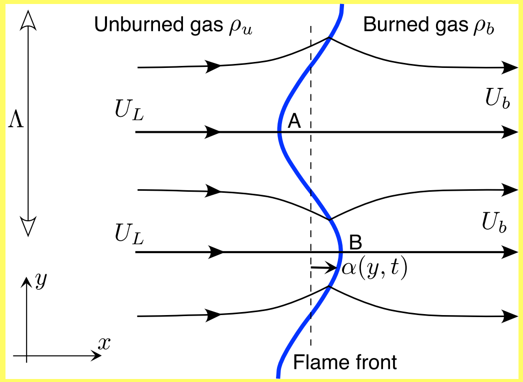



"instantaneous" modification of the flow field, both upstream and donstream
(low Mach nb approx: the speed of sound is infinite, $a \approx \infty$)

tilted front




deviation of the stream lines



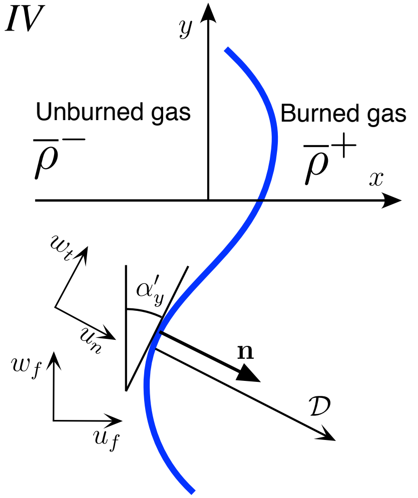

$u_-|_A < U_L$
flame motion

$u_-|_B > U_L$
flame motion

instability mechanism

$$\boxed{\sigma \propto U_L/\Lambda}$$

$$\Lambda \gg d_L, \quad d_L/\Lambda \to 0$$




equation of the perturbed front $\boxed{x = \alpha(y,t)}$ ( reference frame !)

flow velocity at the front $\boxed{\mathbf{u}_f = (u_f,\, w_f)}$

$$\mathbf{n}_f = \left(\frac{1}{\sqrt{1+\alpha_y'^2}},\ -\frac{\alpha'_y}{\sqrt{1+\alpha_y'^2}}\right),\qquad \begin{aligned} u_n \equiv \mathbf{u}_f.\mathbf{n}_f &= (u_f - \alpha'_y w_f)/\sqrt{1+\alpha_y'^2} \\ w_{tg} &= (w_f + \alpha'_y u_f)/\sqrt{1+\alpha_y'^2}\end{aligned}$$

$$\alpha'_y \equiv \partial\alpha/\partial y$$

$$\dot{\alpha}_t \equiv \partial\alpha/\partial t$$

normal velocity of the front

$$\mathcal{D}_f = \frac{\dot{\alpha}_t}{\sqrt{1+\alpha_y'^2}}$$

reference frame of the planar unperturbed flame

$$[x - \alpha(y,t)] \to x \qquad (x = 0)$$

flow velocity relative to the perturbed front

$$U_n \equiv u_n - \mathcal{D}_f = \left(u_f - \dot{\alpha}_t - \alpha'_y w_f\right)/\sqrt{1+\alpha_y'^2} \qquad\qquad W_{tg} = w_{tg}$$

normal component

tangentail component

conservation of mass

$$\overline{\rho}^- U_n^- = \overline{\rho}^+ U_n^+$$

$$\boxed{\overline{\rho}^-\left(u_f^- - \dot{\alpha}_t - \alpha'_y w_f^-\right) = \overline{\rho}^+\left(u_f^+ - \dot{\alpha}_t - \alpha'_y w_f^+\right)}$$

conservation of momentum

$$[p + \overline{\rho} U_n^2]_-^+ = 0 \qquad\qquad [W_{tg}]_-^+ = 0$$

$$\boxed{p_f^- + \overline{\rho}^- \frac{\left(u_f^- - \dot{\alpha}_t - \alpha'_y w_f^-\right)^2}{1+\alpha_y'^2} = p_f^+ + \overline{\rho}^+ \frac{\left(u_f^+ - \dot{\alpha}_t - \alpha'_y w_f^+\right)^2}{1+\alpha_y'^2} \qquad \left(w_f^- + \alpha'_y u_f^-\right) = \left(w_f^+ + \alpha'_y u_f^+\right)}$$

normal

tangential

# Lecture 4 : Hydrodynamic instability of flames

(G. Darrieus 1938, L. Landau 1944)

4-1. Jump across an hydrodynamic discontinuity

4-2. Linearized Euler equations of an incompressible fluid

4-3. Conditions at the front

4-4. Dynamics of passive interfaces

4-5. Darrieus-Landau instability

4-6. Curvature effect: a simplified approach

# **Linearized** Euler equations for an **incompressible** and **inviscid** flow

$$a = \overline{a} + \delta a$$
$$\overline{m}_f = \overline{\rho}^- \overline{u}_f^- = \overline{\rho}^+ \overline{u}_f^+$$

$$\frac{\partial}{\partial x}\delta u^{\pm} + \frac{\partial}{\partial \mathbf{y}}\delta \mathbf{w}^{\pm} = 0,$$
$$\left(\overline{\rho}^{\pm}\frac{\partial}{\partial t} + \overline{m}_f \frac{\partial}{\partial x}\right)\delta u^{\pm} = -\frac{\partial}{\partial x}\delta\pi^{\pm}$$
$$\left(\overline{\rho}^{\pm}\frac{\partial}{\partial t} + \overline{m}_f \frac{\partial}{\partial x}\right)\delta \mathbf{w}^{\pm} = -\frac{\partial}{\partial \mathbf{y}}\delta\pi^{\pm},$$

$\overline{\rho}\, g$ external force per unit volume

$$\pi^{\pm} \equiv p^{\pm} - \overline{\rho}^{\pm} g(t) x,$$

acceleration in the normal direction

$x \to +\infty$ : disturbances remain finite,

$x \to -\infty$ : no disturbances, $\delta u^- = 0$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\delta\pi^{\pm} = 0}$$

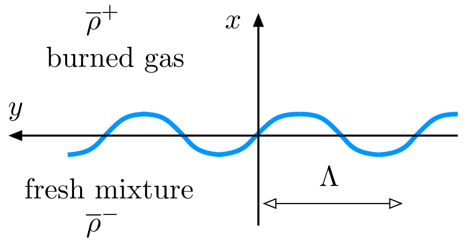
Laplace equation for the pressure

---

## Linear problem

Fourier decomposition of the flame surface $x = \alpha(\mathbf{y}, t)$

$$\alpha(\mathbf{y}, t) = \tilde{\alpha}(t)\, \mathrm{e}^{i\,\mathbf{k}.\mathbf{y}}$$




wave vector $\mathbf{k} = (2\pi/\Lambda)\mathbf{n_k}$

$k = |\mathbf{k}| = 2\pi/\Lambda$

wave length

transverse coordinates

$$\boxed{\delta a(x, \mathbf{y}, t) = \tilde{a}(x, t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}} \qquad \alpha(\mathbf{y}, t) = \tilde{\alpha}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}}}$$

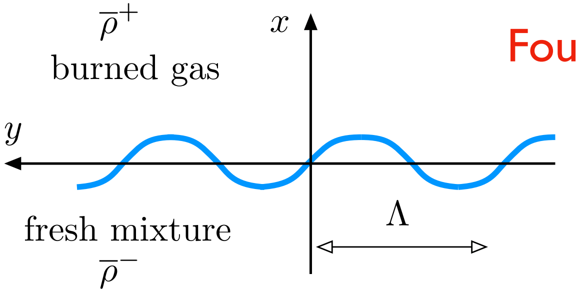
The Fourier components depends also on k

$$\pi^{\pm} \equiv p^{\pm} - \overline{\rho}^{\pm} g(t) x,$$

$$\boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \delta\pi^{\pm} = 0}$$

$x \to +\infty$ : disturbances remain finite,

$x \to -\infty$ : no disturbances, $\quad \delta u^{-} = 0$




Fourier decomposition: $\alpha(\mathbf{y}, t) = \tilde{\alpha}(t)\,\mathrm{e}^{\,i\,\mathbf{k}.\mathbf{y}}$

wave vector $\mathbf{k} = (2\pi/\Lambda)\mathbf{n_k}$ $\qquad k = |\mathbf{k}| = 2\pi/\Lambda$ $\qquad$ given !

$$\boxed{\delta\pi^{\pm}(x, \mathbf{y}, t) = \tilde{\pi}^{\pm}(x, t)\,\mathrm{e}^{i\mathbf{k}.\mathbf{y}}}$$

pressure

$$\frac{\partial^2 \tilde{\pi}^{\pm}}{\partial x^2} - |\mathbf{k}|^2 \tilde{\pi}^{\pm} = 0$$

Laplace equation

$$\boxed{\tilde{\pi}^{\pm}(x, t) = \tilde{\pi}_f^{\pm}(t)\mathrm{e}^{\mp|\mathbf{k}|x}}$$

value at the front $x = 0$

flow velocity

$$\boxed{\delta u^{\pm}(x, \mathbf{y}, t) = \tilde{u}^{\pm}(x, t)\,\mathrm{e}^{i\mathbf{k}.\mathbf{y}} \qquad \delta\mathbf{w}^{\pm}(x, \mathbf{y}, t) = \tilde{\mathbf{w}}^{\pm}(x, t)\,\mathrm{e}^{i\mathbf{k}.\mathbf{y}}}$$

$$\frac{\partial}{\partial x}\delta u^{\pm} + \frac{\partial}{\partial \mathbf{y}}\delta\mathbf{w}^{\pm} = 0, \qquad \left(\overline{\rho}^{\pm}\frac{\partial}{\partial t} + \overline{m}_f\frac{\partial}{\partial x}\right)\delta u^{\pm} = -\frac{\partial}{\partial x}\delta\pi^{\pm}$$

$$\boxed{\frac{\partial \tilde{u}^{\pm}}{\partial x} + \mathrm{i}\mathbf{k}.\tilde{\mathbf{w}}^{\pm} = 0 \qquad \overline{\rho}^{\pm}\left(\frac{\partial}{\partial t} + \overline{u}^{\pm}\frac{\partial}{\partial x}\right)\tilde{u}^{\pm}(x, t) = \pm|\mathbf{k}|\tilde{\pi}_f^{\pm}(t)\mathrm{e}^{\mp|\mathbf{k}|x}}$$

$$\tilde{\pi}^{\pm}(x,t) = \tilde{\pi}_f^{\pm}(t)\mathrm{e}^{\mp|\mathbf{k}|x}$$

$$\frac{\partial \tilde{u}^{\pm}}{\partial x} + \mathrm{i}\mathbf{k}.\tilde{\mathbf{w}}^{\pm} = 0 \qquad \overline{\rho}^{\pm}\left(\frac{\partial}{\partial t} + \overline{u}^{\pm}\frac{\partial}{\partial x}\right)\tilde{u}^{\pm}(x,t) = \pm|\mathbf{k}|\tilde{\pi}_f^{\pm}(t)\mathrm{e}^{\mp|\mathbf{k}|x}$$

general solution to the homogeneous equation + particular solution of the full equation

$$\boxed{\tilde{u}^{\pm}(x,t) = \tilde{u}_R^{\pm}(x,t) + \tilde{u}_P^{\pm}(x,t)}$$

**general solution to the homogeneous equation**

$$\frac{\partial \tilde{u}_R^{\pm}}{\partial t} + \overline{u}^{\pm}\frac{\partial \tilde{u}_R^{\pm}}{\partial x} = 0, \qquad \tilde{u}_R^{\pm} = \tilde{u}_r^{\pm}(t - x/\overline{u}^{\pm}), \qquad \boxed{\tilde{u}_R^{-} = 0, \quad \tilde{u}_R^{+} = \tilde{u}_r^{+}(t - x/\overline{u}^{\pm})}$$

vorticity of the burnt gas flow

$$u^{-}(x,t) = u_P^{-}(x,t)$$

potential flow in unburned gas

**particular solution**

look for a particular solution of the form

source term

$$\tilde{u}_P^{\pm}(x,t) = \tilde{u}_p^{\pm}(t)\mathrm{e}^{\mp kx}, \implies \quad \overline{\rho}^{\pm}\left(\frac{\mathrm{d}}{\mathrm{d}t} \mp \overline{u}^{\pm}k\right)\tilde{u}_p^{\pm}(t) = \pm k\tilde{\pi}_f^{\pm}(t)$$

$$\tilde{\pi}^{\pm}(x,t) = \tilde{\pi}_f^{\pm}(t)\mathrm{e}^{\mp|\mathbf{k}|x}$$

$$x \to -\infty : \tilde{u}^{-} = 0; \quad \tilde{u}^{-}(x,t) = \tilde{u}_f^{-}(t)\mathrm{e}^{kx}$$

3 unknown functions: $\boxed{\tilde{u}_f^{-}(t),\ \tilde{u}_p^{+}(t),\ \tilde{u}_r^{+}(t)}$ $\qquad \tilde{u}_f^{-}(t) \equiv \tilde{u}_p^{-}(t)$

corresponding to the flow at the front $x = 0$

# Linear solution of the Euler equations

3 unknown functions: $\tilde{u}_f^-(t)$, $\tilde{u}_p^+(t)$, $\tilde{u}_r^+(t)$

**flow on the front**

$x < 0$ :

Unburned gas flow

$$\begin{cases} \tilde{u}^-(x,t) = \tilde{u}_f^-(t)\mathrm{e}^{+kx}, \\ k\tilde{\pi}^-(x,t) = -\overline{\rho}^- \left(\dfrac{\mathrm{d}}{\mathrm{d}t} + \overline{u}^- k\right) \tilde{u}_f^-(t)\mathrm{e}^{+kx}, \end{cases}$$

$x > 0$ :

Burned gas flow

$$\begin{cases} \tilde{u}^+(x,t) = \tilde{u}_p^+(t)\mathrm{e}^{-kx} + \tilde{u}_r^+(t - x/\overline{u}^+), \\ k\tilde{\pi}^+(x,t) = \overline{\rho}^+ \left(\dfrac{\mathrm{d}}{\mathrm{d}t} - \overline{u}^+ k\right) \tilde{u}_p^+(t)\mathrm{e}^{-kx}, \end{cases}$$

# Boundary conditions on the front

4 boundary conditions at the flame front involving the additional unknown $\tilde{\alpha}(t)$

2 for the conservation of mass (inner flame structure not modified)

$$\boxed{\delta m_f^- = \delta m_f^+ = 0} \qquad m \equiv \rho(u - \partial\alpha/\partial t)$$

2 for the conservation of normal and tangential momentum

# Lecture 4 : Hydrodynamic instability of flames

4-1. Jump across an hydrodynamic discontinuity

4-2. Linearized Euler equations of an incompressible fluid

4-3. Conditions at the front

conservation of mass

$$\overline{\rho}^- U_n^- = \overline{\rho}^+ U_n^+$$

$$\boxed{\overline{\rho}^- \left(u_f^- - \dot{\alpha}_t - \alpha'_y w_f^-\right) = \overline{\rho}^+ \left(u_f^+ - \dot{\alpha}_t - \alpha'_y w_f^+\right)}$$

conservation of momentum

$$\left[p + \overline{\rho} U_n^2\right]_-^+ = 0 \qquad \left[W_{tg}\right]_-^+ = 0$$

$$p_f^- + \overline{\rho}^- \frac{\left(u_f^- - \dot{\alpha}_t - \alpha'_y w_f^-\right)^2}{1 + \alpha_y'^2} = p_f^+ + \overline{\rho}^+ \frac{\left(u_f^+ - \dot{\alpha}_t - \alpha'_y w_f^+\right)^2}{1 + \alpha_y'^2} \qquad \left(w_f^- + \alpha'_y u_f^-\right) = \left(w_f^+ + \alpha'_y u_f^+\right)$$

normal

tangential

# IV-3) Conditions at the front

notation $a(x,t)$

$a_f(t) \equiv a(x=0,t)$

transverse coordinates

$$\boxed{\delta a(\mathbf{x},\mathbf{y},t) = \tilde{a}(x,t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}} \qquad \alpha(\mathbf{y},t) = \tilde{\alpha}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}}}$$

Mass $\overline{\rho}^-\left(u_f^- - \dot{\alpha}_t - \cancel{\alpha'_y} w_f^-\right) = \overline{\rho}^+\left(u_f^+ - \dot{\alpha}_t - \cancel{\alpha'_y} w_f^+\right) = 0$

$$\boxed{\overline{\rho}^-\left(\delta u_f^- - \dot{\alpha}_t\right) = \overline{\rho}^+\left(\delta u_f^+ - \dot{\alpha}_t\right) = 0} \Rightarrow \delta u_f^- = \delta u_f^+ = \underset{\partial\alpha/\partial t}{\dot{\alpha}_t}$$

$$\tilde{u}_f^-(t) = \tilde{u}_p^+(t) + \tilde{u}_r^+(t) = \mathrm{d}\tilde{\alpha}/\mathrm{d}t$$

potential $\Downarrow$ rotational

$$\frac{\mathrm{d}\tilde{u}_r^+}{\mathrm{d}t} = -\frac{\mathrm{d}\tilde{u}_p^+}{\mathrm{d}t} + \frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2}$$

$\partial/\partial y$ Tangential momentum

$$\boxed{\frac{\partial}{\partial y}\left(w_f^- + \alpha'_y\overline{u}^-\right) = \frac{\partial}{\partial y}\left(w_f^+ + \alpha'_y\overline{u}^+\right)} \Rightarrow k\tilde{u}_p^+(t) + \frac{1}{\overline{u}^+}\frac{\mathrm{d}\tilde{u}_r^+(t)}{\mathrm{d}t} + k\tilde{u}_f^-(t) = \overline{m}_f\left(\frac{1}{\overline{\rho}^+} - \frac{1}{\overline{\rho}^-}\right)k^2\tilde{\alpha}(t)$$

$$\mathrm{i}\mathbf{k}.\tilde{\mathbf{w}}^-(x,t) = -k\tilde{u}^-(x,t), \quad \mathrm{i}\mathbf{k}.\tilde{\mathbf{w}}^+(x,t) = -\frac{\partial}{\partial x}\tilde{u}^+(x,t).$$

$$\partial\alpha'_y/\partial y \to -k^2\tilde{\alpha}(t) \qquad = k\tilde{u}_p^+ + (1/\overline{u}_+)\mathrm{d}\tilde{u}_r^+/\mathrm{d}t$$

elimination of $\mathrm{d}\tilde{u}_r^+/\mathrm{d}t$ $\Downarrow$

$$\boxed{\left(\frac{1}{\overline{u}^+}\frac{\mathrm{d}}{\mathrm{d}t} - k\right)\tilde{u}_p^+ = -\overline{m}_f\left(\frac{1}{\overline{\rho}^+} - \frac{1}{\overline{\rho}^-}\right)k^2\tilde{\alpha} + \frac{1}{\overline{u}^+}\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + k\tilde{u}_f^-}$$

$k\dfrac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}$

Normal momentum

$$\boxed{\delta p_f^- + 2\overline{\rho}^-\overline{u}_f^-(\cancel{\delta u_f^- - \dot{\alpha}_t}) = \delta p_f^+ + 2\overline{\rho}^+\overline{u}_f^+(\cancel{\delta u_f^+ - \dot{\alpha}_t})} \Rightarrow$$

$$\pi^\pm \equiv p^\pm - \overline{\rho}^\pm g(t)x,$$

$$\boxed{\tilde{\pi}_f^+ - \tilde{\pi}_f^- = (\overline{\rho}^- - \overline{\rho}^+)g(t)\tilde{\alpha}(t)}$$

hydrostatic presure

$$x<0: \begin{cases} \tilde{u}^-(x,t) = \tilde{u}_f^-(t)\mathrm{e}^{+kx}, \\ k\tilde{\pi}^-(x,t) = -\overline{\rho}^-\left(\dfrac{\mathrm{d}}{\mathrm{d}t} + \overline{u}^-k\right)\tilde{u}_f^-(t)\mathrm{e}^{+kx}, \end{cases}$$

$$x>0: \begin{cases} \tilde{u}^+(x,t) = \tilde{u}_p^+(t)\mathrm{e}^{-kx} + \tilde{u}_r^+(t - x/\overline{u}^+), \\ k\tilde{\pi}^+(x,t) = \overline{\rho}^+\left(\dfrac{\mathrm{d}}{\mathrm{d}t} - \overline{u}^+k\right)\tilde{u}_p^+(t)\mathrm{e}^{-kx}, \end{cases}$$

$\Downarrow$

$$\Rightarrow \overline{m}_f\left(\frac{1}{\overline{u}^+}\frac{\mathrm{d}}{\mathrm{d}t} - k\right)\tilde{u}_p^+ + \overline{m}_f\left(\frac{1}{\overline{u}^-}\frac{\mathrm{d}}{\mathrm{d}t} + k\right)\tilde{u}_f^- = (\overline{\rho}^- - \overline{\rho}^+)kg(t)\tilde{\alpha}(t)$$

$\dfrac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}$

elimination of $\tilde{u}_p^+$ $\Downarrow$

## Linear dynamical Equation for the front

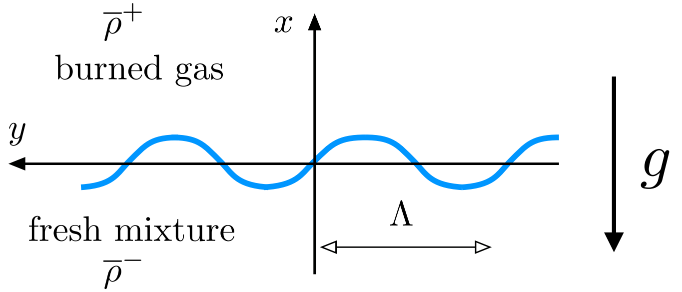
$$\alpha(\mathbf{y},t) = \tilde{\alpha}(t)\,\mathrm{e}^{\,i\,\mathbf{k}.\mathbf{y}}$$

$$\tilde{u}_f^- = \mathrm{d}\tilde{\alpha}/\mathrm{d}t \Rightarrow \boxed{(\overline{\rho}^- + \overline{\rho}^+)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2\overline{m}_f k\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} - \left[(\overline{\rho}^- - \overline{\rho}^+)g(t)k + (\overline{u}^+ - \overline{u}^-)\overline{m}_f k^2\right]\tilde{\alpha} = 0}$$

# SUMMARY

## Equation for the Fourier components of the front

Fourier decomposition of the flame surface $x = \alpha(\mathbf{y}, t)$




$$\alpha(y,t) = \tilde{\alpha}_k(t)\mathrm{e}^{iky}$$

The Fourier components depends also on k

wave vector $\mathbf{k} = (2\pi/\Lambda)\mathbf{n_k}$

wave nb $k = |\mathbf{k}| = 2\pi/\Lambda$

wave length

$$\boxed{(\overline{\rho}^- + \overline{\rho}^+)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2\overline{m}_f k\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} - \left[(\overline{\rho}^- - \overline{\rho}^+)g(t)k + (\overline{u}^+ - \overline{u}^-)\overline{m}_f k^2\right]\tilde{\alpha} = 0}$$

gravity

$$\overline{m}_f = \overline{\rho}\_U_L$$

**Ordinary** differential equation of **second** order for the **Fourier component** of the wrinkle labelled by the wave number

$\tilde{\alpha}_k(t)$

# Lecture 4 : Hydrodynamic instability of flames

4-1. Jump across an hydrodynamic discontinuity

4-2. Linearized Euler equations of an incompressible fluid

4-3. Conditions at the front

4-4. Dynamics of passive interfaces

4-5. Darrieus-Landau instability

4-6. Curvature effect: a simplified approach

*P.Clavin IV*

Rayleigh

# Dynamics of a passive interface $\overline{m}_f = 0$

$$(\overline{\rho}^- + \overline{\rho}^+)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + \cancel{2\overline{m}_f k\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}} - k[(\overline{\rho}^- - \overline{\rho}^+)g(t) + \cancel{(\overline{u}^+ - \overline{u}^-)}\overline{m}_f k]\tilde{\alpha} = 0$$

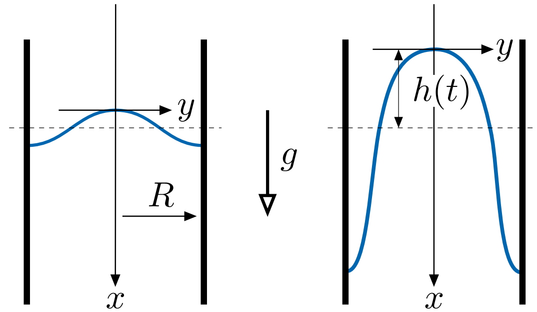
Normal mode analysis $\boxed{\tilde{\alpha}(t) = \hat{\alpha}\,\mathrm{e}^{\sigma t}}$ $\alpha(\mathbf{y}, t) = \tilde{\alpha}(t)\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{y}}$

$\mathrm{Re}(\sigma) < 0:$ linearly stable $\mathrm{Re}(\sigma) > 0:$ linearly unstable

**Rayleigh-Taylor instability**

$g = \text{cst.}$ $(\overline{\rho}^- - \overline{\rho}^+)g > 0$

$$\sigma^2 - A_t kg = 0, \quad A_t > 0$$




Rayleigh-Taylor bubble (upwards propagation)

$g > 0,$ $\mathrm{A_t} \equiv \dfrac{\rho_- - \rho_+}{\rho_- + \rho_+} > 0$ $\sigma = \sqrt{A_t gk}$ $U_{bubble} = 0.361\sqrt{2gR}$

ok with dimension Nonlinear analysis

---

**Gravity waves**

$g = \text{cst.}$ $(\overline{\rho}^- - \overline{\rho}^+)g < 0$

$$\varpi \equiv \mathrm{Im}\,\sigma \neq 0$$

$\tilde{\alpha}(t) = \hat{\alpha}\,\mathrm{e}^{\mathrm{i}\varpi t}$ $\varpi = B\sqrt{gk}$ $B \equiv \sqrt{\dfrac{(\rho_+ - \rho_-)}{(\rho_+ + \rho_-)}}$

Faraday (parametric) instability. Mathieu's equation

$g(t)$ oscillating $$\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + \varpi_o^2\,[1 + \epsilon\cos(\varpi\tau)]\,\tilde{\alpha} = 0$$

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# IV-5) Darrieus-Landau instability of flames

Landau

Darrieus

Darrieus 1938
Landau 1944

$$g = 0 \qquad \overline{m}_f = \overline{\rho}^- \overline{u}^- = \overline{\rho}^+ \overline{u}^+$$

introduce the growth rate

$$(\overline{\rho}^- + \overline{\rho}^+)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2\overline{m}_f k \frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} - (\overline{u}^+ - \overline{u}^-)\overline{m}_f k^2 \tilde{\alpha} = 0$$

$$\boxed{\tilde{\alpha}(t) = \hat{\alpha}\,\mathrm{e}^{\sigma t}}$$

$$\upsilon_b \equiv \overline{\rho}^-/\overline{\rho}^+ = \overline{u}^+/\overline{u}^- > 1 \qquad (\overline{\rho}^- + \overline{\rho}^+)\sigma^2 + 2\overline{m}_f k\sigma - (\overline{u}^+ - \overline{u}^-)\overline{m}_f k^2 = 0$$

$$\overline{u}_- \equiv U_L \qquad \frac{\sigma}{U_L k} = \frac{1}{1+\upsilon_b^{-1}}\left[-1 \pm \sqrt{1+\upsilon_b - \upsilon_b^{-1}}\right]$$

$$\boxed{\sigma = \mathcal{A} U_L k, \quad \mathcal{A} > 0}$$

$d_L/\Lambda \to 0$ : no length scale in the problem; dimensional analysis $\Rightarrow\ \sigma \propto U_L k$

$$\rho_u \gg \rho_b : \ \sigma = \sqrt{U_b U_L}\, k \qquad (\rho_u - \rho_b)/\rho_u \ll 1 : \ \sigma = (U_b - U_L)k/2$$

$k = 2\pi/\Lambda$ shorter is the wavelength stronger is the instability !?
however the analysis is valid only in the limit $d_L/\Lambda \to 0$

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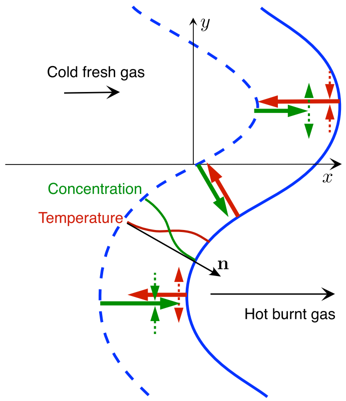
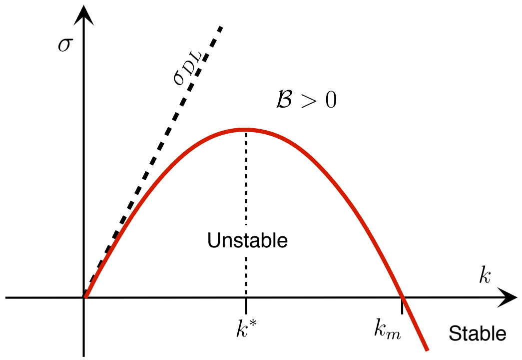



## Stabilisation at small wavelength, $\Lambda \approx d_L$

$$\frac{\partial \alpha}{\partial t} = \mathcal{B} D_T \frac{\partial^2 \alpha}{\partial y^2}. \qquad \sigma_{diff} \equiv 1/\tau_{diff} = -\mathcal{B} D_T k^2 = -\mathcal{B} U_L k (d_L k)$$

first order correction $\quad \mathcal{B} > 0\,?$

$$kd_L < 1 : \quad \boxed{\sigma/U_L = \mathcal{A}k - \mathcal{B}k^2 d_L + \ldots}$$



# Lecture 4 : Hydrodynamic instability of flames

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The full analysis can be found in

**P. Clavin & F.A. Williams** (1982) *Journal of Fluid Mechanics* **116** pp. 251-282

**P. Pelcé & P. Clavin** (1982) *Journal of Fluid Mechanics* **124** pp. 219-237

**P. Clavin & G.Searby** (2016) *Combustion waves and fronts in flows* Cambridge University Press pp. 61-69, 471-473

# Curvature effect: a simplified approach

**G.H. Markestein** (1964) *Nonsteady flame propagation New York: Pergamon*

modification to the inner flame structure $\quad \delta m_f^-(t) \approx \delta m_f^-(t) \neq 0$

first order in perturbation analysis $d_L/\Lambda \ll 1 \quad \delta m_f^-/\overline{\rho}^- \equiv (\delta u_f^- - \dot{\alpha}_t) = -\mathcal{B} D_T \partial^2\alpha/\partial y^2$

$$\tilde{m}_f^-(t)/\overline{m}_f \approx \mathcal{B} d_L k^2 \tilde{\alpha}(t) \qquad D_T = U_L d_L$$

Normal momentum $\quad \delta p_f^- + \boxed{2\overline{\rho}^- \overline{u}_f^-(\delta u_f^- - \dot{\alpha}_t)} = \delta p_f^+ + \boxed{2\overline{\rho}^+ \overline{u}_f^+(\delta u_f^+ - \dot{\alpha}_t)}$

modification of the mass flux introduces new terms in the momentum eq.

(flame notations: $\overline{\rho}^+ \to \rho_b,\ \overline{\rho}^- \to \rho_u,\ \rho_u > \rho_b$) $\quad \tilde{\pi}_{f+} - \tilde{\pi}_{f-} = -2\overline{m}_f \left(\frac{1}{\rho_b} - \frac{1}{\rho_u}\right)\tilde{m}_f(t) + (\rho_u - \rho_b) g(t)\tilde{\alpha}(t)$

equation for the flame front (correction due to curvature, finite thickness effect $kd_L \neq 0$)

$$(\rho_u+\rho_b)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2\overline{m}_f k \frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t}\,(1 + \cancel{\mathcal{B}kd_L}) = k\tilde{\alpha}(\rho_u-\rho_b)\left[g(t) + U_b U_L k\,(1 - 2\mathcal{B}kd_L)\right]$$

curvature effect

flame propagating downwards $g < 0$

$$\frac{1}{k_m} \equiv 2\mathcal{B}d_L \qquad \boxed{\left(1 + \frac{\rho_b}{\rho_u}\right)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2U_L k\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} = \left(\frac{\rho_u}{\rho_b} - 1\right)k\left[-\frac{\rho_b}{\rho_u}|g| + U_L^2 k\left(1 - \frac{k}{k_m}\right)\right]\tilde{\alpha}}$$

non-dimensional parameters $\quad \upsilon_b \equiv \overline{\rho}^-/\overline{\rho}^+ = \overline{u}^+/\overline{u}^- > 1 \quad$ Froude number

$s = \sigma\tau_L \quad \kappa \equiv kd_L \quad \kappa_m \equiv 1/(2\mathcal{B}) \quad \mathcal{G}_0 \equiv (\rho_b/\rho_u)\mathrm{Fr}^{-1} \quad \mathrm{Fr}^{-1} \equiv |g|d_L/U_L^2$

$$(1 + \upsilon_b^{-1})s^2 + 2\kappa s - (\upsilon_b - 1)\kappa\left[-\mathcal{G}_o + \kappa\left(1 - \frac{\kappa}{\kappa_m}\right)\right] = 0$$

## Stability limits of flames propagating downwards

marginal wavenumber
$s \equiv \sigma\tau_L = 0$

$$\boxed{\left[-\mathcal{G}_o + \kappa\left(1 - \frac{\kappa}{\kappa_m}\right)\right] = 0,}$$

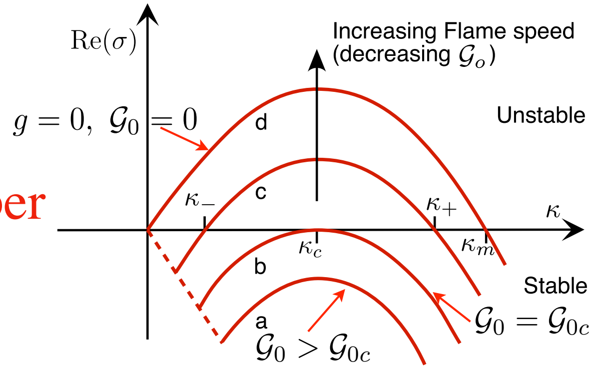
# Stability limits of flames propagating downwards

non-dimensional parameters $\kappa \equiv kd_L$ $\kappa_m \equiv 1/(2\mathcal{B})$ $\mathcal{G}_0 \equiv (\rho_b/\rho_u)\mathrm{Fr}^{-1}$ $\mathrm{Fr}^{-1} \equiv |g|d_L/U_L^2$

$s = \sigma\tau_L$ $$(1+v_b^{-1})s^2 + 2\kappa s - (v_b - 1)\kappa\left[-\mathcal{G}_o + \kappa\left(1 - \frac{\kappa}{\kappa_m}\right)\right] = 0$$

marginal wavenumber $\sigma = 0$ $$\boxed{\left[-\mathcal{G}_o + \kappa\left(1 - \frac{\kappa}{\kappa_m}\right)\right] = 0,}$$

quadratic equation for the marginal wave number




gravity stabilizes the large wavelengths of slow propagating flame

curvature stabilizes the small wavelengths

the planar flame is stable (at all wavelength) beyond a stability threshold, namely for slow flames $U_L < 10\mathrm{cm/s}$

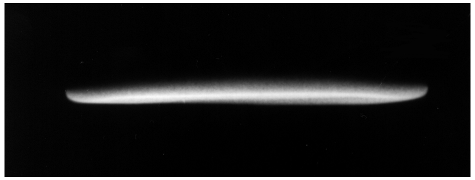
Marginal wave number $k_c$ **at the instability threshold** $U_L \approx 10\mathrm{cm/s}$

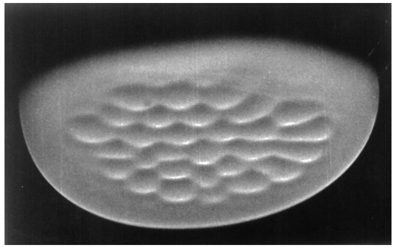
$$\mathcal{G}_{oc} = \frac{k_c d_L}{2}, \quad k_c = \frac{k_m}{2}, \quad U_{Lc} = \sqrt{2\frac{\rho_b}{\rho_u}\frac{|g|}{k_c}}$$

OK with experiments

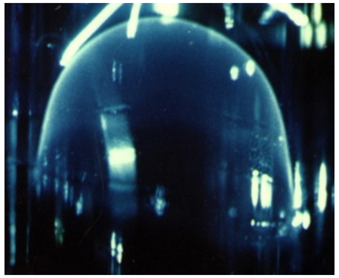
**G. Searby and J. Quinard** (1990) *Combust. Flame*, 83 (3-4) 298-311

**Slow** flame propagating **downwards**





Flames propagating upwards: bubble flames



*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

## Lecture V
## Thermo-diffusive phenomena

# Lecture 5: Thermo-diffusive phenomena

## 5-1. Flame stretch and Markstein numbers

*Passive interfaces*

*One-step flame model*

*The second Markstein number*

## 5-2. Thermo-diffusive instabilities

$\mathrm{Le} \equiv D_T/D$

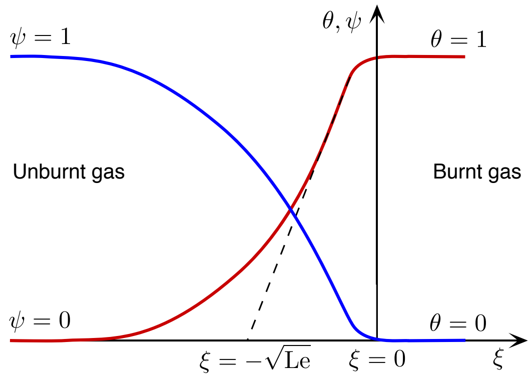
*Planar flames for* $\mathrm{Le} \neq 1$

*Jump conditions across the reaction layer*

*Linear equations and linear analysis*

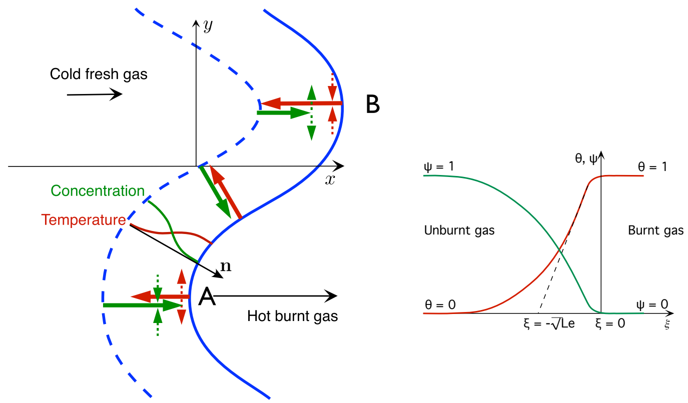
*Cellular instability* $(\mathrm{Le} < 1)$

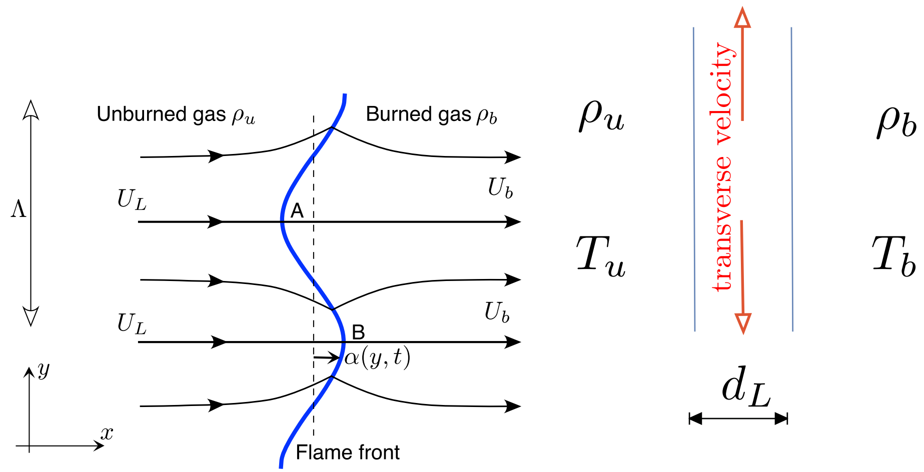
*Oscillatory instability* $(\mathrm{Le} > 1)$



# 5-1. Flame stretch and Markstein numbers

## Two mechanisms modify the inner flame structure




transverse diffusion

transverse convection

One-step model $\quad R \rightarrow P + Q \quad \beta \gg 1$

A single scalar: the Markstein number $\mathcal{M}$

$$\boxed{(U_n^- - U_L)/U_L = -\mathcal{M}(\tau_L/\tau_s)}$$

$U_n^-$ normal flame velocity in the fresh mixture

$1/\tau_s$ rate of stretch of flame surface

$$U_n^- = u_n^- - \mathcal{D}_f, \quad u_n^- \equiv \mathbf{n}_f.\mathbf{u}_f^-$$

# Stretch rate, strain and curvature of a flame

## Passive interface (0 thickness, fully convected )

element of surface area $\delta^2 s$ $\qquad \dfrac{1}{\tau_s} = \dfrac{1}{\delta^2 s}\dfrac{\mathrm{d}\delta^2 s}{\mathrm{d}t}$ $\qquad \mathrm{d}\mathbf{r}_f/\mathrm{d}t = \mathbf{u}^e(\mathbf{r}_f)$

element of volume $\qquad \delta^3 r = \delta^2 s\, \delta\zeta$

thickness of the element

$$\frac{1}{\delta^3 r}\frac{\mathrm{d}}{\mathrm{d}t}\delta^3 r = \nabla.\mathbf{u}^e|_f$$

continuity

coordinate normal to the front $\zeta$

$$\frac{1}{\delta^3 r}\frac{\mathrm{d}}{\mathrm{d}t}\delta^3 r = \frac{1}{\delta^2 s}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 s + \frac{1}{\delta\zeta}\frac{\mathrm{d}}{\mathrm{d}t}\delta\zeta = \nabla.\mathbf{u}^e|_f$$

$|\mathbf{n}_f| = 1$ normal to the front

$$\mathrm{d}\delta\zeta/\mathrm{d}t = \mathbf{n}_f.\left[\mathbf{u}^e(\mathbf{r}_f + \delta\zeta\,\mathbf{n}_f) - \mathbf{u}^e(\mathbf{r}_f)\right]$$

flow velocity

$$\delta\zeta \ll 1: \ \mathbf{u}^e\,(\mathbf{r}_f + \delta\zeta\,\mathbf{n}_f) \approx \mathbf{u}^e\,(\mathbf{r}_f) + \delta\zeta\ \mathbf{n}_f.\nabla\mathbf{u}^e$$

$$\frac{1}{\delta\zeta}\frac{\mathrm{d}}{\mathrm{d}t}\delta\zeta = \ \mathbf{n}_f.\nabla\mathbf{u}^e.\mathbf{n}_f$$

$$\boxed{\frac{1}{\tau_s} \equiv \frac{1}{\delta^2 s}\frac{\mathrm{d}}{\mathrm{d}t}\delta^2 s = \nabla.\mathbf{u}^e|_f - \mathbf{n}_f.\nabla\mathbf{u}^e|_f.\mathbf{n}_f.}$$

## Flame (first order correction) $d_L/\Lambda \ll 1$

flame **moving** with the **normal** velocity $U_L$

$\mathbf{n}_f.\mathbf{n}_f = 1$ $\qquad$ flow velocity relative to the flame $\mathbf{u}^e(\mathbf{r}_f) = \mathbf{u}_f^- - U_L\mathbf{n}_f$

flow velocity in the laboratory frame

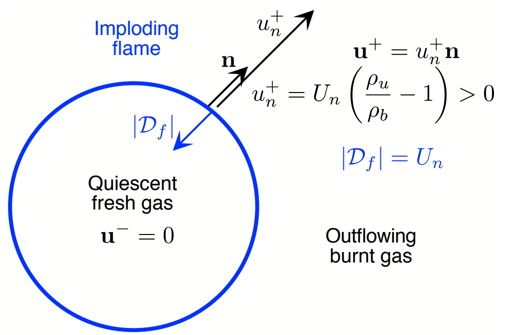
$\mathbf{n}_f.\nabla\mathbf{n}|_f.\mathbf{n}_f = 0$

$$1/\tau_s = -U_L\nabla.\mathbf{n}_f + \nabla.\mathbf{u}^-|_f - \mathbf{n}_f.\nabla\mathbf{u}^-|_f.\mathbf{n}_f$$

$$1/\tau_s = -U_L \nabla.\mathbf{n}_f + \cancel{\nabla.\mathbf{u}^-|_f} - \mathbf{n}_f.\nabla\mathbf{u}^-|_f.\mathbf{n}_f$$

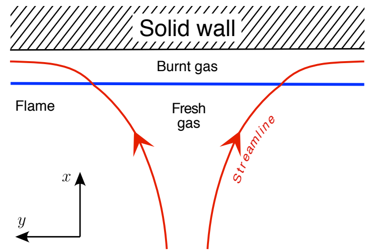
incompressibility

differential geometry $-\nabla.\mathbf{n}_f = 1/R \equiv (1/R_1 + 1/R_2)$




$$\boxed{1/\tau_s = U_L/R - \mathbf{n}_f.\nabla\mathbf{u}^-|_f.\mathbf{n}_f}$$

front curvature strain rate




# First order correction to the laminar flame velocity $R \rightarrow P + Q$

$$\tau_L/\tau_s \ll 1, \quad \tau_L \equiv d_L/U_L, \quad (U_n^- - U_L)/U_L \propto \tau_L/\tau_s$$

*Clavin, Williams, JFM (1982)* ***116*** *p. 252-282,* *Clavin, Garcia, J. Méc. Théor. Appl. (1983)* ***2****(2) p. 245-263,*

Lewis number $\mathrm{Le} \equiv \mathrm{D}_T/\mathrm{D}$ Asymptotic analysis: $\beta \gg 1$ $\beta(\mathrm{Le} - 1) = O(1)$

$$\boxed{(U_n^- - U_L)/U_L = -\mathcal{M}(\tau_L/\tau_s)}$$

$\upsilon_b \equiv \rho_u/\rho_b > 1$ $\theta \equiv (T - T_u)/(T_b - T_u)$ $l \equiv \beta(\mathrm{Le} - 1)$ heat conductivity $\lambda(\theta)$

gas expansion $\Rightarrow$ hydrodynamics kinetics + diffusion

$$\boxed{\mathcal{M} = \frac{\upsilon_b}{\upsilon_b - 1}\mathcal{J} + \frac{l}{2}\frac{\mathcal{D}}{(\upsilon_b - 1)}}$$

$$\mathcal{J} = \int_0^1 \frac{(\upsilon_b - 1)\lambda(\theta)}{1 + (\upsilon_b - 1)\theta}\mathrm{d}\theta, \quad \mathcal{D} = -\int_0^1 \frac{(\upsilon_b - 1)\lambda(\theta)\ln\theta}{1 + (\upsilon_b - 1)\theta}\mathrm{d}\theta,$$

*P.Clavin V*

# The second Markstein number

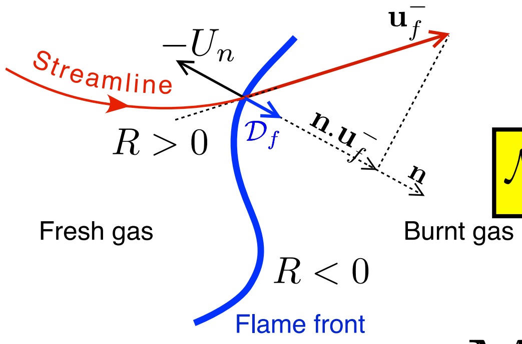
## multiple-step flame model

*Clavin, Graña-Otero, JFM (2011)* ***689*** *p. 187-217,*

$$(U_n^- - U_L)/U_L = -\mathcal{M}_{fc}(d_L/R) + \mathcal{M}_{sr}(\tau_L \mathbf{n}_f.\nabla\mathbf{u}^-|_f.\mathbf{n}_f)$$

front curvature flow strain rate

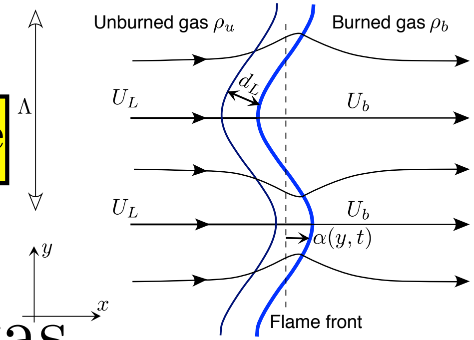
$$\boxed{\mathcal{M}_{fc} \neq \mathcal{M}_{sr}}$$



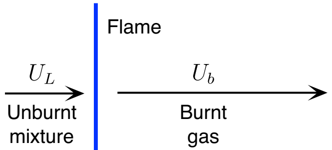

difficulty with the finite thickness:

$$\boxed{\mathcal{M}_{sr} \text{ varies with the position inside the flame structure}}$$

$$\mathcal{M}_{sr}(\tau_L \mathbf{n}_f.\nabla\mathbf{u}^-|_f.\mathbf{n}_f) = \text{cst.}$$




# Markstein numbers in the burned gas

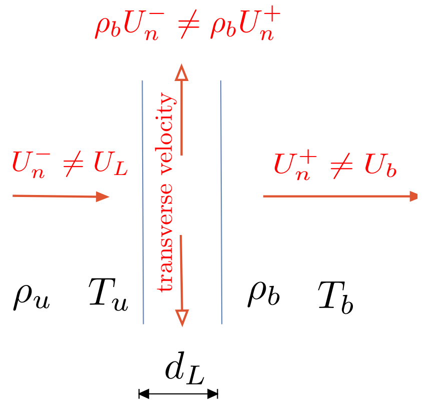



$$\frac{U_b}{U_L} = \frac{T_b}{T_u}$$

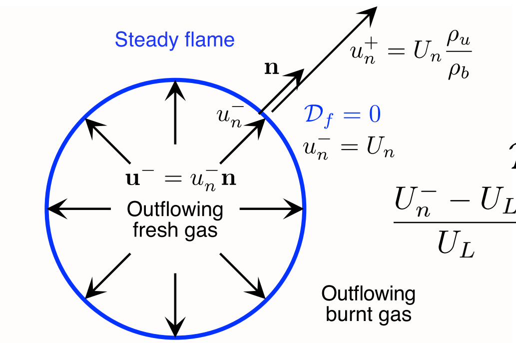
$$U_n^+ \equiv u_n^+ - \mathcal{D}_f \qquad u_n^+ \equiv \mathbf{n}_f.\mathbf{u}^+(\mathbf{r}_f)$$

$$\frac{(U_n^+ - U_b)}{U_b} = -\mathcal{M}_{fc}^+ \frac{d_L}{R} + \mathcal{M}_{sr}^+ \tau_L \mathbf{n}_f.\nabla\mathbf{u}^+|_f.\mathbf{n}_f$$

$$\boxed{\mathcal{M}_{fc}^+ \neq \mathcal{M}_{fc}^- \qquad \mathcal{M}_{sr}^+ \neq \mathcal{M}_{sr}^-}$$




numerical and experimental data

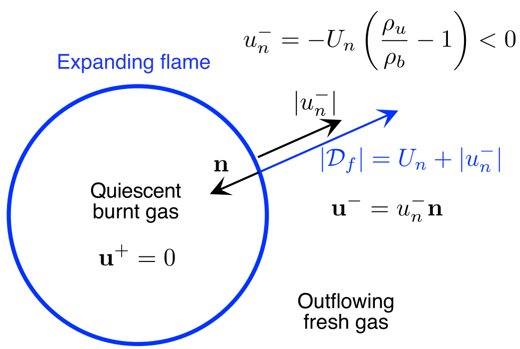



$$\mathcal{D}_f = 0, \quad U_n^- = u_n^-$$

$$\frac{U_n^- - U_L}{U_L} = -2(\mathcal{M}_{fc}^- - \mathcal{M}_{sr}^-)\frac{d_L}{R_f}$$

$$u_n^+ = 0 \quad U_n^+ = -\mathcal{D}_f$$

$$\frac{U_n^+ - U_b}{U_b} = -2\mathcal{M}_{fc}^+ \frac{d_L}{R_f}$$



# 5-2. Thermo-diffusive instabilities

instability mechanism $\neq$ hydrodynamic instability

equations

$$\rho T = \rho_o T_o \qquad \rho c_p \mathrm{D}T/\mathrm{D}t = \nabla.(\lambda \nabla T) + \sum_j Q^{(j)} \dot{W}^{(j)}(T, ..Y_k..)$$

$$\rho \mathrm{D}Y_i/\mathrm{D}t = \nabla.(\rho D_i \nabla Y_i) + \sum_j \vartheta_i^{(j)} \mathsf{m}_i \dot{W}^{(j)}(T, ..Y_k..),$$

+ fluid ~~mechanics~~

**Thermo-diffusive** flame **model** for a one-step kinetics

$$\theta \equiv \frac{T - T_u}{T_b - T_u} \in [0,1] \qquad \psi \equiv Y/Y_u \qquad \beta \equiv \frac{E}{k_B T_b}\left(1 - \frac{T_u}{T_b}\right) \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

$\rho =$ cst.

$\rho u = 0$ in the lab. frame

$$\boxed{\begin{array}{ll} \dfrac{\partial \theta}{\partial t} - D_T \Delta \theta = \dfrac{\psi}{\tau_{rb}} \mathrm{e}^{-\beta(1-\theta)} & \dfrac{\partial \psi}{\partial t} - D \Delta \psi = -\dfrac{\psi}{\tau_{rb}} \mathrm{e}^{-\beta(1-\theta)} \\ x = -\infty : \theta = 0, \psi = 1 & x = +\infty : \theta = 1, \psi = 0 \end{array}}$$

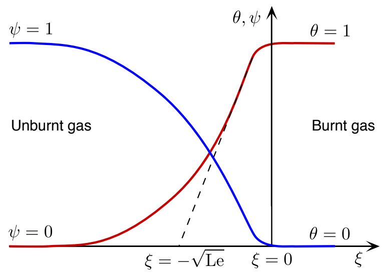
Unperturbed (steady state) **planar flame** Le$\neq 1$

(frame attached to the flame front)

$$\xi \equiv \frac{x - \partial\alpha/\partial t}{d_L(\mathrm{Le}{=}1)}$$

$$\mu \equiv \frac{U_L}{U_L(\mathrm{Le} = 1)} \quad ?$$

$$\mu \frac{\mathrm{d}\theta}{\mathrm{d}\xi} - \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = \frac{\beta^2}{2} \psi \mathrm{e}^{-\beta(1-\theta)} \qquad \mu \frac{\mathrm{d}\psi}{\mathrm{d}\xi} - \frac{1}{\mathrm{Le}} \frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = -\frac{\beta^2}{2} \psi \mathrm{e}^{-\beta(1-\theta)}$$




reaction layer

$$-\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} \approx \frac{\beta^2}{2} \psi \mathrm{e}^{-\beta(1-\theta)} \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \frac{1}{\mathrm{Le}} \frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = 0$$

matching

$$\boxed{\mu = \sqrt{\mathrm{Le}}}$$

(first order reaction rate)

**reduced laminar flame velocity**

## Mathematical model for analyzing the thermo-diffusive instability Le≠1:

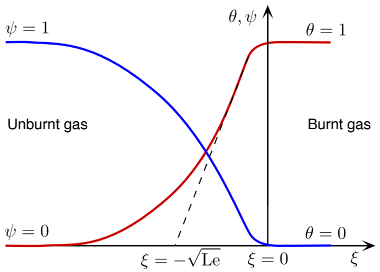
$$\theta \equiv \frac{T - T_u}{T_b - T_u} \in [0,1] \qquad \psi \equiv Y/Y_u \qquad \beta \equiv \frac{E}{k_B T_b}\left(1 - \frac{T_u}{T_b}\right) \qquad \frac{1}{\tau_{rb}} \equiv \frac{\mathrm{e}^{-E/k_B T_b}}{\tau_{coll}}$$

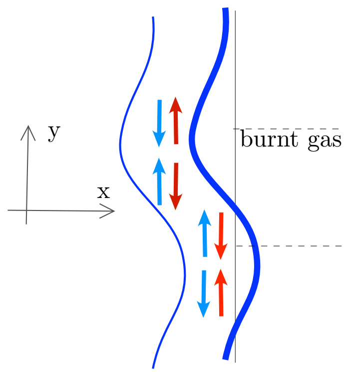
$$\boxed{\begin{array}{ll} \dfrac{\partial \theta}{\partial t} - D_T \Delta\theta = \dfrac{\psi}{\tau_{rb}}\mathrm{e}^{-\beta(1-\theta)} & \dfrac{\partial \psi}{\partial t} - D\Delta\psi = -\dfrac{\psi}{\tau_{rb}}\mathrm{e}^{-\beta(1-\theta)} \\ x = -\infty : \theta = 0, \psi = 1 & x = +\infty : \theta = 1, \psi = 0 \end{array}}$$

Lewis number Le $\equiv \mathrm{D}_T/\mathrm{D}$ Asymptotic analysis: $\beta \gg 1 \quad \beta(\mathrm{Le} - 1) = O(1)$

*Sivashinsky, Combust. Sci. Technol.* (1977) **15** p. 137-146,

*Joulin, Clavin, Combust. Flame* (1979) **35** p. 139-153




competition of transverse diffusive fluxes of species and temperature for the flame temperature on the reaction sheet

shifted longitudinal profiles

# Flame temperature of curved flame for $\mathrm{Le} \neq 1$

$$\mathrm{Le} \neq 1 \quad \Rightarrow \quad \theta_f \neq 1$$

perturbed flame temperature (reaction layer)

$$\beta \gg 1 \quad \boxed{\mathrm{Le} - 1 = O(1/\beta) \quad \Rightarrow \quad (\theta_f - 1) = O(1/\beta)} \qquad \theta_f \equiv \frac{T_f - T_u}{T_b - T_u} \; ?$$

unperturbed flame temperature

# the thin reaction layer of curved flame is quasi-planar

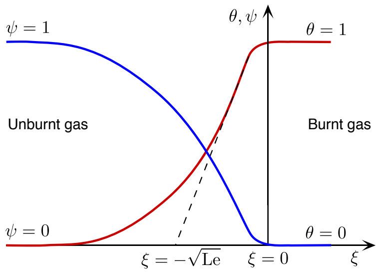
$$-\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} \approx \frac{\beta^2}{2}\psi \mathrm{e}^{-\beta(1-\theta)} \qquad \boxed{\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \frac{1}{\mathrm{Le}}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = 0}$$

$\xi =$ non-dimensional normal coordinate $\quad \xi \equiv \dfrac{x}{d_L(\mathrm{Le}=1)}$

$$\theta = \theta_f - \Theta_1/\beta + .. \qquad \psi = -\Psi_1/\beta + .. \qquad \Theta_1 = \Psi_1/\mathrm{Le} \qquad \frac{1}{\beta^2}\frac{\mathrm{d}^2\Theta_1}{\mathrm{d}\xi^2} = \frac{1}{2}\mathrm{e}^{-\beta(1-\theta_f)}\Psi_1 \mathrm{e}^{-\Theta_1}$$

$\beta(\theta_f - 1) = O(1) \qquad \mathrm{d}\theta/\mathrm{d}\xi|_{0+} = O(1/\beta) \qquad$ integration and matching $\quad \mathrm{d}\theta/\mathrm{d}\xi|_{\xi=0-} \approx \mathrm{Le}^{1/2}\,\mathrm{e}^{-\beta(1-\theta_f)/2}$

# jump conditions across the reaction layer




$$\mathrm{d}\theta/\mathrm{d}\xi|_{\xi=0-} = e^{-\beta(1-\theta_f)/2} \qquad \left[\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \frac{1}{\mathrm{Le}}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right]_{0-}^{0+} = 0$$

valid at the leading order $\qquad$ valid up to $1^{st}$ order

*Joulin, Clavin; Combust. Flame* (1979) ***35*** *p. 139-153*
*Clavin, Searby; Cambrige University Press* (2014) *p. 390-393*

# Preheated zone

non-dimensional equations in the reference frame attached to the unperturbed flame

$$\xi \equiv x/d_L, \quad \eta = y/d_L, \quad \tau \equiv t/\tau_L \qquad \frac{\partial\theta}{\partial\tau} + \frac{\partial\theta}{\partial\xi} - \Delta\theta = 0, \quad \frac{\partial\psi}{\partial\tau} + \frac{\partial\psi}{\partial\xi} - \frac{1}{\mathrm{Le}}\Delta\psi = 0$$

boundary conditions: jump conditions and $\quad \xi = -\infty \;: \theta = 0, \psi = 1, \qquad \xi = \infty \;: \theta = 1, \psi = 0.$

# Linear equations

frame of reference attached to the reaction sheet $(\zeta, \eta, \tau)$

reduced coordinates:
reference state=steady flame $d_L, \tau_L$

reduced equation for the wrinkled reaction sheet: $\xi = \mathrm{a}(\eta, \tau)$

$$\zeta \equiv \xi - \mathrm{a}(\eta, \tau) \qquad \boxed{\zeta = 0 : \text{reaction sheet}}$$

change of variable: $\dfrac{\partial}{\partial \xi} = \dfrac{\partial}{\partial \zeta}, \quad \dfrac{\partial}{\partial \eta} \to \dfrac{\partial}{\partial \eta} - \dfrac{\partial \mathrm{a}}{\partial \eta}\dfrac{\partial}{\partial \zeta}, \quad \dfrac{\partial}{\partial \tau} \to \dfrac{\partial}{\partial \tau} - \dfrac{\partial \mathrm{a}}{\partial \tau}\dfrac{\partial}{\partial \zeta}$

linearization $\quad \theta = \overline{\theta}(\zeta) + \delta\theta, \quad \theta_f = 1 + \delta\theta_f, \quad \psi = \overline{\psi}(\zeta) + \delta\psi$

external equations

$$\left[\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \zeta} - \left(\frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2}\right)\right] \delta\theta = \left(\frac{\partial \mathrm{a}}{\partial \tau} - \frac{\partial^2 \mathrm{a}}{\partial \eta^2}\right) \frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\zeta}$$

$$\left[\frac{\partial}{\partial \tau} + \frac{\partial}{\partial \zeta} - \frac{1}{\mathrm{Le}}\left(\frac{\partial^2}{\partial \zeta^2} + \frac{\partial^2}{\partial \eta^2}\right)\right] \delta\psi = \left(\frac{\partial \mathrm{a}}{\partial \tau} - \frac{1}{\mathrm{Le}}\frac{\partial^2 \mathrm{a}}{\partial \eta^2}\right) \frac{\mathrm{d}\overline{\psi}}{\mathrm{d}\zeta}$$

harmonic analysis
(normal modes)

$$\mathrm{a}(\eta, \tau) = \hat{a}\mathrm{e}^{(\mathrm{i}\kappa\eta + \varsigma\tau)}, \quad \delta\theta_f(\eta, \tau) = \tilde{\theta}_f \hat{a}\mathrm{e}^{(\mathrm{i}\kappa\eta + \varsigma\tau)}$$

$\varsigma \equiv \sigma\tau_L, \quad \kappa \equiv k d_L$

$$\delta\psi = \tilde{\psi}(\zeta)\hat{a}\mathrm{e}^{(\mathrm{i}\kappa\eta + \varsigma\tau)} \qquad \delta\theta = \tilde{\theta}(\zeta)\hat{a}\mathrm{e}^{(\mathrm{i}\kappa\eta + \varsigma\tau)}$$

complex number
unknown

real number
given

$\varsigma(\kappa)$?
reduced linear growth rate

$$\left[\frac{\mathrm{d}}{\mathrm{d}\zeta} - \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right] \tilde{\theta}(\zeta) + \left(\varsigma + \kappa^2\right) \tilde{\theta}(\zeta) = \left(\varsigma + \kappa^2\right) \frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\zeta}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\zeta} - \frac{1}{\mathrm{Le}}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right] \tilde{\psi}(\zeta) + \left(\varsigma + \frac{\kappa^2}{\mathrm{Le}}\right) \tilde{\psi}(\zeta) = \left(\varsigma + \frac{\kappa^2}{\mathrm{Le}}\right) \frac{\mathrm{d}\overline{\psi}}{\mathrm{d}\zeta}$$

boundary conditions: jump conditions and $\zeta = -\infty \; : \tilde{\theta} = 0, \tilde{\psi} = 0, \qquad \zeta = +\infty \; : \tilde{\theta} = 0, \tilde{\psi} = 0$

external equations

$$\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}-\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right]\tilde{\theta}(\zeta)+\left(\varsigma+\kappa^2\right)\tilde{\theta}(\zeta)=\left(\varsigma+\kappa^2\right)\frac{\mathrm{d}\overline{\theta}}{\mathrm{d}\zeta}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}-\frac{1}{\mathrm{Le}}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\right]\tilde{\psi}(\zeta)+\left(\varsigma+\frac{\kappa^2}{\mathrm{Le}}\right)\tilde{\psi}(\zeta)=\left(\varsigma+\frac{\kappa^2}{\mathrm{Le}}\right)\frac{\mathrm{d}\overline{\psi}}{\mathrm{d}\zeta}$$

$\kappa$ given

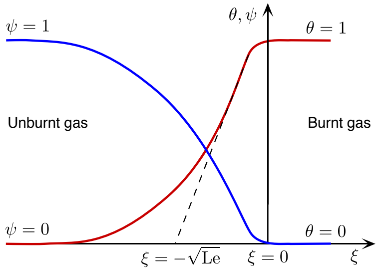
$\varsigma$ and $\theta_f$ ?

## Linear solutions in the external zones (preheated and burned gas)

particular solutions $\quad \tilde{\theta}=\mathrm{d}\overline{\theta}/\mathrm{d}\zeta \qquad \tilde{\psi}=\mathrm{d}\overline{\psi}/\mathrm{d}\zeta$

$$\overline{\theta}^-=\mathrm{e}^{\zeta} \quad \overline{\theta}^+=1 \qquad \overline{\psi}^-=1-\mathrm{e}^{\mathrm{Le}\zeta} \quad \overline{\psi}^+=0$$

boundary conditions




$$\zeta\rightarrow\pm\infty: \quad \tilde{\theta}=0$$
$$\theta(\zeta=0)=\theta_f$$

$$\tilde{\theta}^{\pm}=\mathrm{d}\overline{\theta}^{\pm}/\mathrm{d}\zeta+\left(\tilde{\theta}_f-\mathrm{d}\overline{\theta}^{\pm}/\mathrm{d}\zeta\Big|_{\zeta=0}\right)\mathrm{e}^{r^{\pm}\zeta}$$

general solution to the homogeneous equation

$$r^2-r-\left(\varsigma+\kappa^2\right)=0 \qquad r^{\pm}=\frac{1}{2}\left[1\mp\sqrt{1+4(\varsigma+\kappa^2)}\right]$$

mass fraction in the preheated zone zone

$$\zeta\rightarrow-\infty: \quad \tilde{\psi}=0$$
$$\psi(\zeta=0)=0$$

$$\tilde{\psi}^-=\mathrm{d}\overline{\psi}^-/\mathrm{d}\zeta-\left(\mathrm{d}\overline{\psi}^-/\mathrm{d}\zeta\Big|_{\zeta=0}\right)\mathrm{e}^{s^-\zeta}$$

burned gas:

$$\zeta>0: \quad \tilde{\psi}=\overline{\psi}=0$$

$$\frac{1}{\mathrm{Le}}s^2-s-\left(\zeta+\frac{\kappa^2}{\mathrm{Le}}\right)=0 \qquad s^-=\frac{\mathrm{Le}}{2}\left[1+\sqrt{1+\frac{4}{\mathrm{Le}}\left(\varsigma+\frac{\kappa^2}{\mathrm{Le}}\right)}\right]$$

jump conditions

$$\boxed{\left.d\theta/d\xi\right|_{\xi=0-} = e^{-\beta(1-\theta_f)/2} \quad \text{valid at the leading order} \qquad \left[\frac{d\theta}{d\xi} + \frac{1}{\text{Le}}\frac{d\psi}{d\xi}\right]_{0-}^{0+} = 0 \quad \text{valid up to } 1^{st} \text{ order}}$$

$\varsigma$ and $\theta_f$ ?

asymptotic analysis $\beta \gg 1:$ $\quad \text{Le} = 1 + l/\beta, \quad l \equiv \beta(\text{Le}-1) = O(1)$

$$\beta(1-\theta_f) = O(1) \qquad \beta\tilde{\theta}_f = O(1)$$

$2^{nd}$ condition $\quad \tilde{\theta}_f(r^+ - r^-) = (s^- - r^-) + (1 - \text{Le})$

$$(r^+ - r^-) = -\sqrt{1+4(\varsigma+\kappa^2)}$$

$$\text{Le} \to 1: \quad s^- - r^- \to 0$$

to leading order in small values of (Le-1) $= O(1/\beta)$

$$\boxed{\tilde{\theta}_f \approx \frac{(\text{Le}-1)}{2}\left[\frac{1}{\sqrt{1+4(\varsigma+\kappa^2)}} - 1 + \frac{2\varsigma + 4\kappa^2}{1+4(\varsigma+\kappa^2)}\right]}$$

linearized $1^{st}$condition $\quad \left.d\tilde{\theta}^-/d\zeta\right|_{\zeta=0} = \beta\tilde{\theta}_f/2 \qquad 1 - r^- = r^+ = \beta\tilde{\theta}_f/2$

valid to leading order using $\tilde{\theta}_f = O(1/\beta)$

$$\boxed{\beta\tilde{\theta}_f = 1 - \sqrt{1+4(\varsigma+\kappa^2)}}$$

dispersion relation $\quad \zeta(\kappa)$

root of

$$\boxed{-\frac{l}{2}\left[1 - \sqrt{1+4(\varsigma+\kappa^2)} + 2\varsigma\right] = \left[1 - \sqrt{1+4(\varsigma+\kappa^2)}\right]\left[1+4(\varsigma+\kappa^2)\right]}$$

$$-\frac{l}{2}\left[1-\sqrt{1+4(\varsigma+\kappa^2)}+2\varsigma\right]=\left[1-\sqrt{1+4(\varsigma+\kappa^2)}\right]\left[1+4(\varsigma+\kappa^2)\right]$$

$\mathrm{e}^{\sigma t}=\mathrm{e}^{\varsigma\tau}$ linear instability: $\mathrm{Re}\,\varsigma>0\quad(\mathrm{Re}\,\sigma>0)$

Cellular instability for $\mathrm{Le}\equiv D_T/D<1$

---

Weakly curved limit. Small wavenumber expansion $\kappa\equiv kd_L\ll 1$

$$\kappa=0:\varsigma(\kappa)=0\qquad|\varsigma|\equiv|\sigma|\tau_L\ll 1$$

$$\tilde{\theta}_f=(\mathrm{Le}-1)\kappa^2\qquad \boxed{\varsigma=-(l+2)\kappa^2/2}\quad \sigma\equiv\zeta/\tau_L$$ is real

$l\equiv\beta(\mathrm{Le}-1)>-2$ : $\sigma<0$ stable

$l\equiv\beta(\mathrm{Le}-1)<-2$ : $\sigma>0$ unstable $\mathcal{M}<0$ $\quad\dfrac{\partial\alpha}{\partial t}\propto -D_T\dfrac{\partial^2\alpha}{\partial y^2}$

$$\boxed{\frac{\partial\alpha}{\partial t}=\left[\frac{\beta(\mathrm{Le}-1)+2}{2}\right]D_T\frac{\partial^2\alpha}{\partial y^2}}$$

$$\frac{\partial\alpha}{\partial t}=\left[\frac{\beta(\mathrm{Le}-1)+2}{2}\right]D_T\left(\frac{\partial^2\alpha}{\partial y^2}+\frac{\partial^2\alpha}{\partial z^2}\right)=\left[\beta(\mathrm{Le}-1)+2\right]\frac{D_T}{R},\qquad 2/R=1/R_1+1/R_2$$

$$\mathcal{M}=\beta(\mathrm{Le}-1)+2\qquad \mathcal{M}=\frac{\upsilon_b}{\upsilon_b-1}\mathcal{J}+\frac{l}{2}\frac{\mathcal{D}}{(\upsilon_b-1)}$$

$$\mathcal{J}=\int_0^1\frac{(\upsilon_b-1)\lambda(\theta)}{1+(\upsilon_b-1)\theta}\mathrm{d}\theta,\quad \mathcal{D}=-\int_0^1\frac{(\upsilon_b-1)\lambda(\theta)\ln\theta}{1+(\upsilon_b-1)\theta}\mathrm{d}\theta,$$

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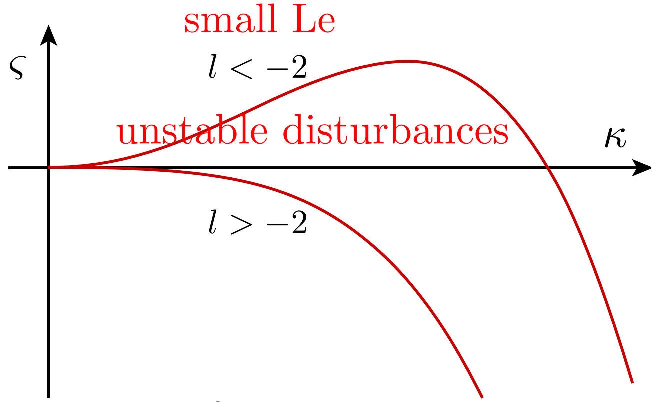
$$\varsigma=-(l+2)\kappa^2/2-8\kappa^4$$

Turing type of instability

$\mathrm{Re}\,\varsigma>0\qquad \mathrm{Im}\,\varsigma=0$




Zeldovich

Turing

heavy hydrocarbons: $D_F < D_{O_2}$

lean mixtures of an heavy hydrocarbon $\Rightarrow$ limiting species $= F$
species in excess $= O_2$

$$D = D_F \qquad\qquad D_T = D_{O_2} > D_F$$

$$\mathrm{Le} \equiv D_T/D = D_{O_2}/D_F > 1 \quad \Rightarrow$$
$$\beta(\mathrm{Le} - 1) > -2 \quad \Rightarrow$$

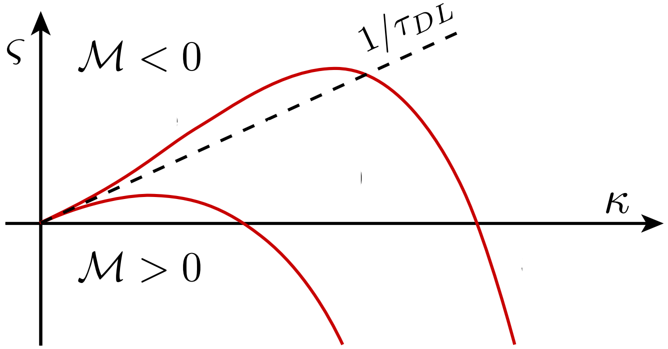
thermo-diffusive **stable**

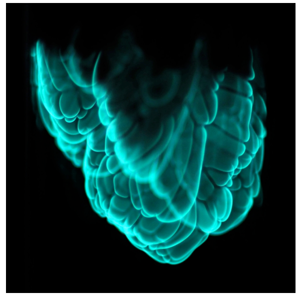
rich mixtures of heavy hydrocarbons are thermo-diffusive **unstable**
example: propane-air

rich mixtures of light fuels are thermo-diffusive **stable**
example: hydrogen-air

# Hydrodynamics + diffusion

Propane lean flame
$\mathcal{M} > 0$
hydrodynamic instability only



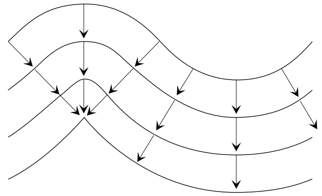



Propane rich flame
$\mathcal{M} < 0$
hydrodynamic + cellular instabilities
shorter wavelengths are unstable

Sivashinsky eq. 1977 $\dfrac{\partial \phi}{\partial \tau} - \mathcal{H}(\phi) - \triangle\phi + \dfrac{1}{2}|\nabla\phi|^2 = 0$

nonlinear equation (weak gas expansion)

geometrical term: Huygens construction



*Sivashinsky, Acta. Astronaut.* (1977) **4** p. 1177-1206

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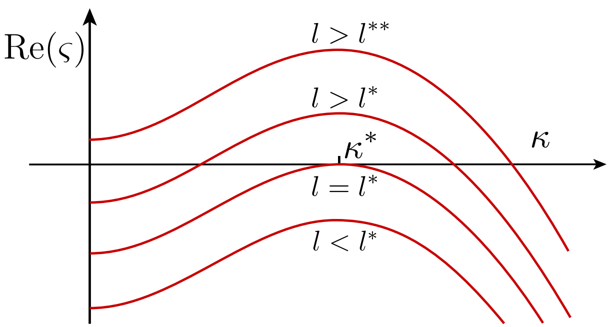
# Oscillatory instability $\mathrm{Le} \equiv D_T/D > 1$

$\mathrm{Im}(\varsigma) \neq 0$

$l \equiv \beta(\mathrm{Le}-1) = l^* :\ \mathrm{Re}(\varsigma) = 0,\ \kappa^* \neq 0$

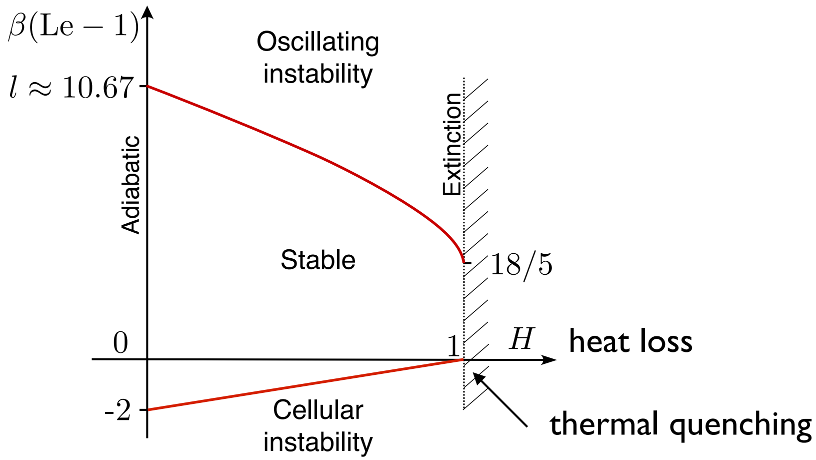
Poincaré-Andronov bifurcation $l^* \approx 10$




$l^{**} \approx 11$ : planar pulsation. OK for solid combustion

effect of heat losses

*Joulin, Clavin, Combust. Flame* (1979) **35** p. 139-153



*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS


## Lecture VI
## Thermal quenching of flames and flammability limits

# Lecture 6: Thermal quenching and flammability limits

6-1. Extinction through thermal loss

6-2. Basic concepts in chemical kinetics

*Combustion of hydrogen*

*Two-step model. Crossover temperature*

*One-step model with temperature cutoff*

6-3. Flame speed near flammability limits

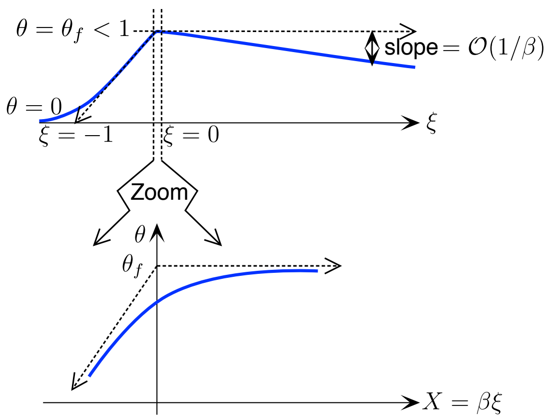
# 6-1. Extinction through thermal loss

Davy 1830

Zeldovich 1941

## a small heat loss can quench the flame

Formulation (volumetric heat loss in a planar flame)




$$\mu \frac{\mathrm{d}\theta}{\mathrm{d}\xi} - \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = w - \frac{\tau_L}{\tau_{\text{cool}}}\theta, \qquad \mu \frac{\mathrm{d}\psi}{\mathrm{d}\xi} - \frac{1}{\text{Le}}\frac{\mathrm{d}^2\psi}{\mathrm{d}\xi^2} = -w$$

$$\xi \equiv x/d_L \qquad \mu = U_L/U_{Ladia} \qquad 1/\tau_{cool} \approx D_T/R^2$$

$$\tau_L \approx D_T/U_L^2 \quad \Rightarrow \quad \frac{\tau_L}{\tau_{cool}} \approx \left(\frac{D_T}{R\,U_L}\right)^2 \qquad R = \text{tube radius}$$

$$\xi = -\infty \;:\; \theta = 0,\; \psi = 1, \qquad \xi = +\infty \;:\; \theta = 0,\; \psi = 0$$

## Asymptotic analysis for small heat loss and a one-step reaction

*Joulin, Clavin, Combust. Flame* (1979) **35** p. 139-153

$$\beta \to \infty \qquad \boxed{\tau_L/\tau_{cool} = h/\beta} \qquad h = O(1) \qquad \boxed{\beta(1-\theta_f) = O(1)} \qquad w(\theta,\psi) = (\beta^2/2)\psi\exp[-\beta(1-\theta)]$$

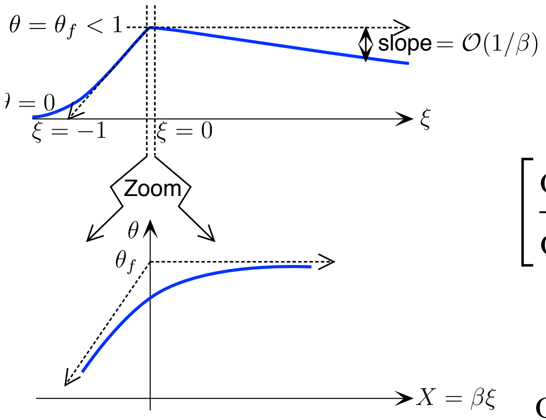
unknown flame temperature < adiabatic flame temperature : $\theta_f < 1$

external solutions : $w = 0$

$$\xi < 0 : \begin{cases} \theta_-(\xi) & = \theta_f \mathrm{e}^{[\mu + h/(\beta\mu)]\xi}, \\ \psi_-(\xi) & = 1 - \mathrm{e}^{\text{Le}\,\mu\xi}, \end{cases} \qquad \xi > 0 : \begin{cases} \theta_+(\xi) & = \theta_f \mathrm{e}^{-[h/(\beta\mu)]\xi}, \\ \psi_+(\xi) & = 0, \end{cases}$$

jumps across the thin reaction zone :

$$\mathrm{d}\theta/\mathrm{d}\xi|_{\xi=0-} = e^{-\beta(1-\theta_f)/2} \qquad \left[\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \frac{1}{\text{Le}}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right]_{0-}^{0+} = 0$$




$$\xi < 0 : \begin{cases} \theta_-(\xi) & = \theta_f \mathrm{e}^{[\mu + h/(\beta\mu)]\xi}, \\ \psi_-(\xi) & = 1 - \mathrm{e}^{\mathrm{Le}\,\mu\xi}, \end{cases} \qquad \xi > 0 : \begin{cases} \theta_+(\xi) & = \theta_f \mathrm{e}^{-[h/(\beta\mu)]\xi}, \\ \psi_+(\xi) & = 0, \end{cases}$$

$$\left[\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \frac{1}{\mathrm{Le}}\frac{\mathrm{d}\psi}{\mathrm{d}\xi}\right]_{0-}^{0+} = 0 \qquad \Rightarrow \quad -(h/\beta\mu)\theta_f - (\mu + h/\beta\mu)\theta_f + \mu = 0$$

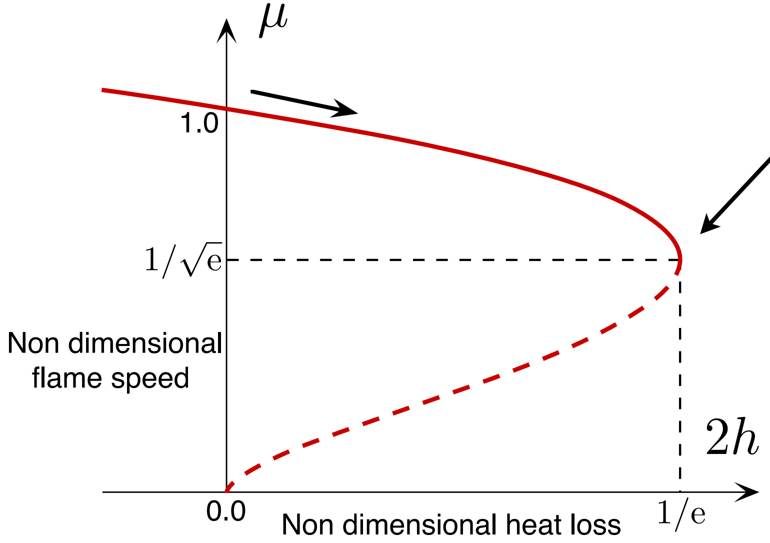
$$\theta_f - 1 = O(1/\beta) \;\Rightarrow\; \boxed{\beta(1-\theta_f) = 2h/\mu^2}$$

$$\mathrm{d}\theta/\mathrm{d}\xi|_{\xi=0-} = e^{-\beta(1-\theta_f)/2} \quad \Rightarrow \quad \mu = \exp(-h/\mu^2)$$

$$\boxed{\mu^2 \ln \mu^2 = -2h}$$

C-shaped curve: no solution for $2\,h \;>\; 1/\mathrm{e}$

quenching at finite flame velocity $U_L/U_{Ladia} = 1/\sqrt{\mathrm{e}}$



# Lecture 6: Thermal quenching and flammability limits

6-1. Extinction through thermal loss

6-2. Basic concepts in chemical kinetics

*Combustion of hydrogen*

*Two-step model. Crossover temperature*

*One-step model with temperature cutoff*

6-3. Flame speed near flammability limits

# 6-2. Basic concepts in chemical kinetics

*C.K. Law; Cambridge University Press* (2006)

## Combustion of hydrogen

units: moles/cm$^3$, s$^{-1}$ and Kelvin

$$dc_{ij}/dt = -\omega_j$$

$$\omega_j = \tilde{k}_j c_{1j} c_{2j} \quad \text{or} \quad \omega_j = \tilde{k}_j c_{1j} c_{2j} c_{3j}$$

$$\tilde{k}_j = \tilde{B}_j T^{\nu} e^{-T_{aj}/T}$$

| Label | Reaction | $\tilde{k}_j$ | $\tilde{B}_j$ | $\nu_j$ | $T_{aj}$ |
|---|---|---|---|---|---|
| 1 | $O_2 + H \rightleftharpoons OH + O$ | $\tilde{k}_{1f}$ | $3.52 \times 10^{16}$ | -0.7 | 8590 |
| | | $\tilde{k}_{1b}$ | $7.04 \times 10^{13}$ | -0.264 | 72 |
| 2 | $H_2 + OH \rightleftharpoons H_2O + H$ | $\tilde{k}_{2f}$ | $1.17 \times 10^{9}$ | 1.3 | 1825 |
| | | $\tilde{k}_{2b}$ | $1.29 \times 10^{10}$ | 1.196 | 9412 |
| 3 | $H_2 + O \rightleftharpoons OH + H$ | $\tilde{k}_{3f}$ | $5.06 \times 10^{4}$ | 2.67 | 3165 |
| | | $\tilde{k}_{3b}$ | $3.03 \times 10^{4}$ | 2.63 | 2433 |
| 4f | $O_2 + H + M \rightarrow HO_2 + M$ | $\tilde{k}_{4f}$ | $5.79 \times 10^{19}$ | -1.4 | 0 |
| 5f | $H + H + M \rightarrow H_2 + M$ | $\tilde{k}_{5f}$ | $1.30 \times 10^{18}$ | -1 | 0 |
| 6f | $H + OH + M \rightarrow H_2O + M$ | $\tilde{k}_{6f}$ | $4.00 \times 10^{22}$ | -2 | 0 |
| 7f | $HO_2 + H \rightarrow OH + OH$ | $\tilde{k}_{7f}$ | $7.08 \times 10^{13}$ | 0 | 148 |
| 8f | $HO_2 + H \rightarrow H_2 + O_2$ | $\tilde{k}_{8f}$ | $1.66 \times 10^{13}$ | 0 | 414 |
| 9f | $HO_2 + OH \rightarrow H_2O + O_2$ | $\tilde{k}_{9f}$ | $2.89 \times 10^{13}$ | 0 | -250 |

shuffle reactions
(1f), (2f), (3f)
chain branching

$$\boxed{\begin{array}{c} O_2 + 3H_2 \rightarrow 2H_2O + 2H \\ \text{rate: } \omega_{1f} \equiv c_H c_{O_2} k_{1f} \end{array}}$$

(4f) chain breaking

$$\boxed{\begin{array}{c} M + H + O_2 \rightarrow M + HO_2, \\ \text{rate: } \omega_{4f} = c_H c_{O_2} n k_{4f} \end{array}}$$

$$k_{4f} = B_{4f}$$

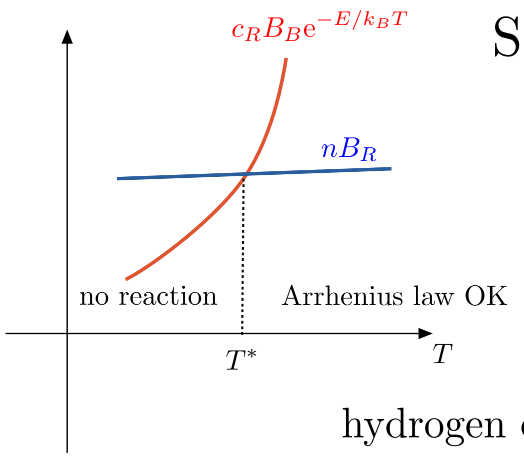
(8b) initiation

$$H_2 + O_2 \rightarrow HO_2 + H$$

*Sanchez & Williams, Prog. Energy Combust. Sci.* (2014) ***41*** *p. 1-55*

## Simplified two-step model: crossover temperature

*Clavin, Searby; Cambridge University Press* (2014) *p. 390-393*




$$\text{(B)} \quad R + X \rightarrow 2X, \quad \omega_B = c_R c_X B_B e^{-E/k_B T}, \quad E \gg k_B T$$

$$\text{(R)} \quad M + X \rightarrow P + Q, \quad \omega_R = c_X n B_R, \quad E_R = 0$$

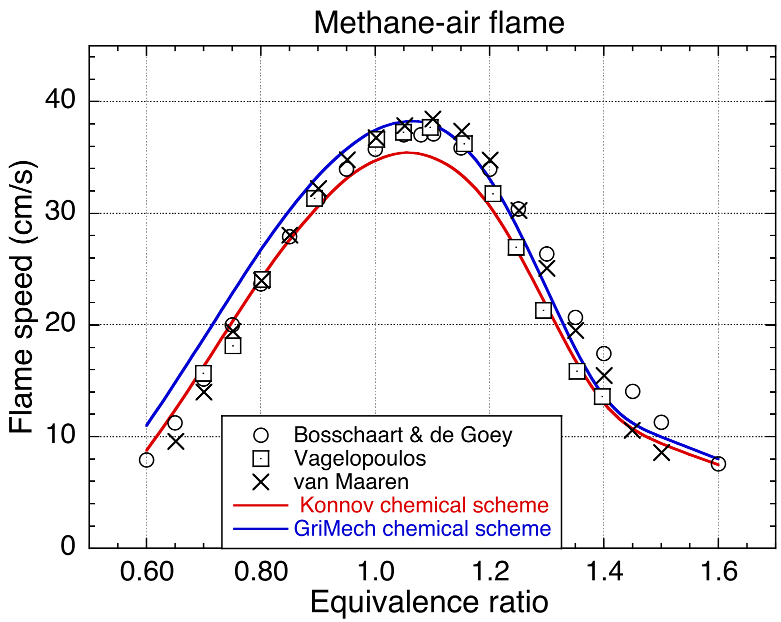
$$c_R B_B e^{-E_1/k_B T^*} = n B_R \qquad T^* \in [900K - 1400K]$$

hydrogen combustion: $k_{1f}(T^*) \equiv \boxed{B_{1f} e^{-E_1/k_B T^*} = n B_{4f}}$

## Flammability limit

$$T_b = T^* \Rightarrow \boxed{q_R Y_u^* \equiv c_p (T^* - T_u)}$$

# Methane-air flame




$$\vartheta_F^+ F + \vartheta_o^+ O_2 \rightleftharpoons P$$

Equivalence ratio

$$\phi = \frac{N_F/N_{O_2}}{\vartheta_F^+/\vartheta_{O_2}^+}$$

$\phi = 1:$ stoichiometry

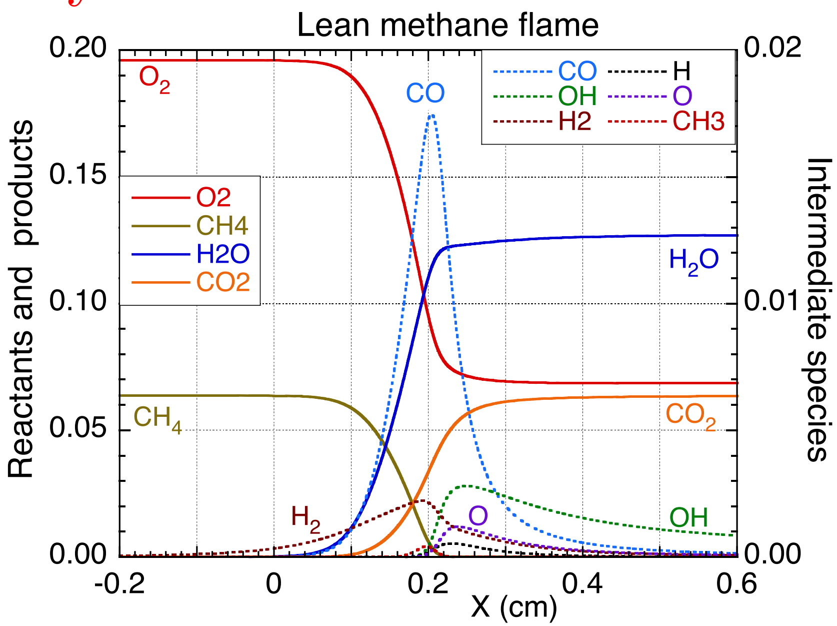
$\phi > 1:$ fuel rich

$\phi < 1:$ fuel lean

near to the flammability limit

$\phi = 0.65$

"thicker flame"



# Two-step model for rich hydrogen flames near the flammability limit

(consumption of hydroperoxide included)

| Label | Reaction | $\tilde{k}_j$ | $\tilde{B}_j$ | $\nu_j$ | $T_{aj}$ |
|---|---|---|---|---|---|
| 1 | $O_2 + H \rightleftharpoons OH + O$ | $\tilde{k}_{1f}$ | $3.52 \times 10^{16}$ | -0.7 | 8590 |
| | | $\tilde{k}_{1b}$ | $7.04 \times 10^{13}$ | -0.264 | 72 |
| 2 | $H_2 + OH \rightleftharpoons H_2O + H$ | $\tilde{k}_{2f}$ | $1.17 \times 10^{9}$ | 1.3 | 1825 |
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| 3 | $H_2 + O \rightleftharpoons OH + H$ | $\tilde{k}_{3f}$ | $5.06 \times 10^{4}$ | 2.67 | 3165 |
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| 5f | $H + H + M \rightarrow H_2 + M$ | $\tilde{k}_{5f}$ | $1.30 \times 10^{18}$ | -1 | 0 |
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| 8f | $HO_2 + H \rightarrow H_2 + O_2$ | $\tilde{k}_{8f}$ | $1.66 \times 10^{13}$ | 0 | 414 |
| 9f | $HO_2 + OH \rightarrow H_2O + O_2$ | $\tilde{k}_{9f}$ | $2.89 \times 10^{13}$ | 0 | -250 |

$$O_2 + 3H_2 \rightarrow 2H_2O + 2H,$$

$$\omega_{1f} = c_H c_{O_2} k_{1f}(T), \quad k_{1f}(T) = B_{1f} e^{-E_1/k_B T}$$

$$H + H \rightarrow H_2 + Q,$$

$$\omega_{4f} + \omega_{5f} = n c_H c_{O_2} B_{4f} + \boxed{n c_H^2 B_{5f}}$$

$$\boxed{\frac{dc_H}{dt} = \left[B_{1f} e^{-E_1/k_B T} - n B_{4f}\right] c_{O_2} c_H - n B_{5f} c_H^2}$$

$\boxed{(B_{1f} e^{-E_1/k_B T} - n B_{4f})/nB_{4f} \ll 1}$ tri molecular recombination reaction $\boxed{\text{(5f)}}$ $\Rightarrow$ $\boxed{H \text{ in quasi-steady state}}$

$$T > T^* : \boxed{c_H \approx c_{O_2} \frac{[B_{1f} e^{-E_1/k_B T} - n B_{4f}]}{n B_{5f}}} \qquad T < T^* : \quad c_H = 0$$

## One-step model (near the flammability limit)

$$c_H B_5 \ll c_{O_2} B_{4f} \quad \Rightarrow \quad \omega_{5f} \ll \omega_{4f}$$

$$n B_{4f} = B_{1f} e^{-E_1/k_B T^*} \qquad 1/\tau^* \equiv (n B_{4f}^2 c^*_{O_2 u})/B_{5f}$$

$$m \frac{d\theta}{dx} - \rho D_T \frac{d^2\theta}{dx^2} \approx \frac{\rho}{\tau^*} \psi^2 J(T)$$

$$m \frac{d\psi}{dx} - \rho D_{O_2} \frac{d^2\psi}{dx^2} \approx -\frac{\rho}{\tau^*} \psi^2 e^{-\frac{E}{k_B}(\frac{1}{T} - \frac{1}{T^*})} J(T)$$

$$\begin{cases} T > T^* : & J(T) \equiv \dfrac{T_u}{T}[e^{-\frac{E}{k_B}(\frac{1}{T} - \frac{1}{T^*})} - 1] \\ T < T^* : & J(T) = 0 \end{cases}$$

$$x = -\infty : \ \theta = 0, \qquad x = +\infty : \ \theta = 1$$

reaction of order 2 with a temperature cutoff

very close to the flammability limit $\quad \dfrac{T_b - T^*}{T^*} \ll \dfrac{k_B T^*}{E} \quad \Rightarrow \quad [e^{-\frac{E}{k_B}(\frac{1}{T} - \frac{1}{T^*})} - 1] \approx \dfrac{E}{k_B}(\dfrac{1}{T^*} - \dfrac{1}{T}) \ll 1$

# 6-3. Flame speed near flammability limits

*He, Clavin, Combust. Flame (1993)* **93** *p. 391-408*

$$\theta \equiv \frac{(T - T_u)}{(T_b - T_u)} \in \boxed{[\theta^*, 1]} \qquad \theta^* \equiv \frac{(T^* - T_u)}{(T_b - T_u)} \qquad T_b > T^* \;\Rightarrow \boxed{\theta^* < 1 \text{ but close to } 1}$$

$$m\frac{\mathrm{d}\theta}{\mathrm{d}x} - \rho_b D_T \frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \approx \frac{\rho_b}{\tau^*}\psi^2 j(\theta)$$

$$m\frac{\mathrm{d}\psi}{\mathrm{d}x} - \rho_b D_{O_2} \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} \approx -\frac{\rho_b}{\tau^*}\psi^2 j(\theta)$$

$$\begin{cases} \theta > \theta^* : & j(\theta) \approx b^*(\theta - \theta^*) \\ \theta < \theta^* : & j(\theta) = 0 \end{cases} \qquad b^* \equiv \frac{T_u}{T^*}\frac{E}{k_B T^*}\frac{T_b - T_u}{T^*}$$

reaction zone: $\psi = \mathrm{Le}(1-\theta)$, $\quad D_T \dfrac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = \dfrac{\mathrm{Le}^2 b^*}{\tau^*}(1-\theta)^2[(\theta-1)-(\theta^*-1)]$ $\qquad \mathrm{Le} \equiv D_T/D_{O_2}$

$$\times\frac{\mathrm{d}\theta}{\mathrm{d}x} + \int_{\theta^*}^{1} \mathrm{d}\theta + \text{matching} \;\Rightarrow\; D_T \left.\frac{\mathrm{d}\theta}{\mathrm{d}x}\right|_- \approx \mathrm{Le}\sqrt{\frac{b^*}{6}}(1-\theta^*)^2\sqrt{\frac{D_T}{\tau^*}} \qquad \frac{\rho_u}{\rho_b}\frac{U_L}{\sqrt{D_T/\tau^*}} \approx \mathrm{Le}\sqrt{\frac{b^*}{6}}(1-\theta^*)^2$$

$$0 < \frac{T_b - T^*}{T_b - T_u} \ll 1 \quad \Rightarrow \quad \boxed{\frac{\rho_u}{\rho^*}\frac{U_L}{\sqrt{D_T/\tau^*}} \approx \mathrm{Le}\sqrt{\frac{b^*}{6}}\left(\frac{T_b - T^*}{T^* - T_u}\right)^2}$$

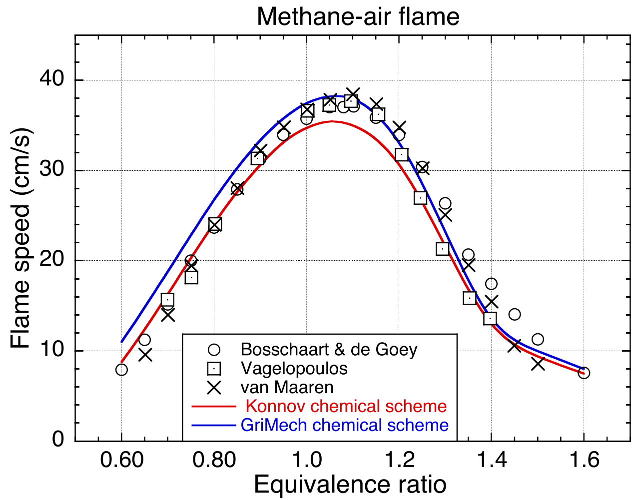
the flame velocity decreases smoothly to zero when approaching the flammability limit $T_b \to T^*$

the flame thickness $d_L^*$ diverges, $T_b \to T^*$ : $\dfrac{d_L^*}{d_L} \propto \dfrac{1}{\beta^2}\left(\dfrac{T^* - T_u}{T_b - T^*}\right)^2$

## Divergence of the thermal sensitivity: Thermal quenching

$$\frac{T_b}{U_L}\frac{\mathrm{d}U_L}{\mathrm{d}T_b} = \frac{2T_b}{T_b - T^*} \nearrow \infty$$

the least heat loss quenches the flame at a non zero velocity



*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

## Lecture VII
## Flame kernels and quasi-isobaric ignition

# Lecture 7: Flame kernels and quasi-isobaric ignition

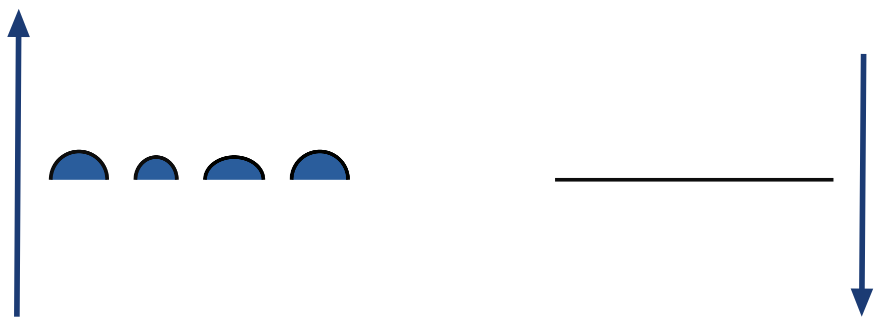
# Introduction

**Flammability limits** **Critical conditions of ignition**

Some hydrocarbon lean mixtures that are flammable cannot be ignited quasi-isobarically

Some hydrocarbon rich mixtures that are non-flammable can sustain curved flames

**Upward propagation limit is different from downwards**



**Ignition in turbulent flows**

Princeton experiments (2014) Wu & al.

Turbulence facilitates ignition of hydrocarbon lean mixtures

Turbulence may suppress ignition of hydrocarbon rich mixtures

# Lecture 7: Flame kernels and quasi-isobaric ignition

# VII-2) Zeldovich critical radius

Zeldovich

Flame kernel for a flame far from the flammability limits

Unstable **steady spherical solution** for the one-step model of adiabatic flames

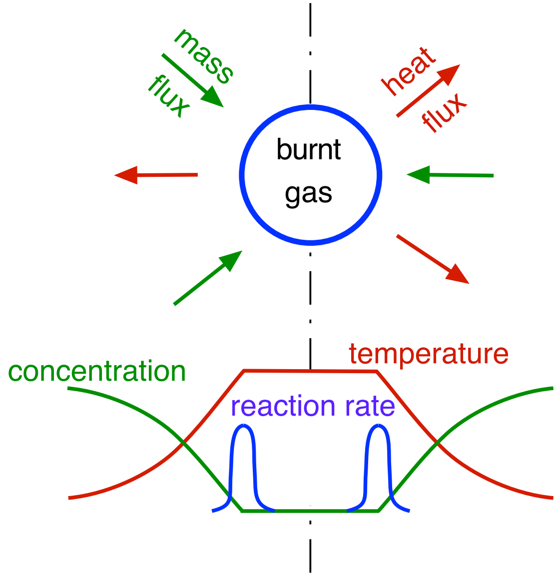
$$\boxed{\theta \equiv (T - T_u)/(T_b - T_u) \qquad \psi \equiv Y/Y_u} \qquad \tau_{rb} \equiv \tau_r(T_b) \qquad c_p(T_b - T_u) = q_R Y_u$$

$$\Delta\theta = \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\theta}{\mathrm{d}R}\right) \qquad \Delta\psi = \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\psi}{\mathrm{d}R}\right)$$

No flow

$$-D_T\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\theta}{\mathrm{d}R}\right) = D\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\psi}{\mathrm{d}R}\right) = \frac{\psi}{\tau_{rb}}\mathrm{e}^{-\beta(1-\theta)}$$

$$R \leqslant R_f: \quad \theta = \theta_f, \quad \psi = 0; \qquad R \to \infty: \quad \theta = 0, \quad \psi = 1$$




Flame temperature

$$\mathrm{Le} \neq 1 \;\Rightarrow\; T_f \neq T_b$$

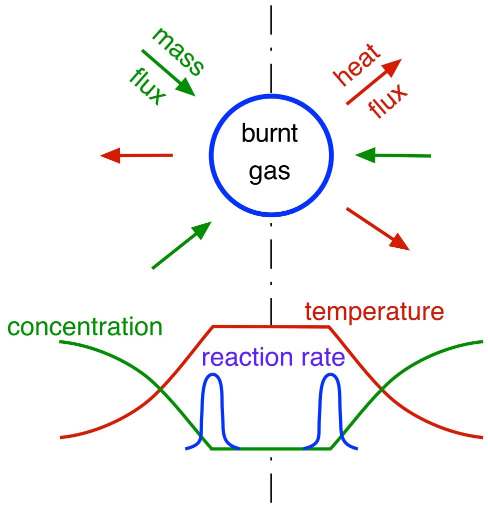
$$D_T\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\theta}{\mathrm{d}R}\right) + D\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\psi}{\mathrm{d}R}\right) = 0 \qquad \text{(conservation energy)}$$

double integration from $R = 0$ to $R = \infty$

$$D_T\theta = D(1-\psi) \quad \Rightarrow \quad \boxed{\theta_f = 1/\mathrm{Le}}$$

$$\boxed{\begin{array}{l} \mathrm{Le} \equiv D_T/D \\ \mathrm{Le} < 1 \Rightarrow T_f > T_b \\ \mathrm{Le} > 1 \Rightarrow T_f < T_b \end{array}}$$

 Unstable **steady spherical solution** for the one-step model of adiabatic flames




$$\theta \equiv (T - T_u)/(T_b - T_u) \qquad \psi \equiv Y/Y_u \qquad \tau_{rb} \equiv \tau_r(T_b) \qquad c_p(T_b - T_u) = q_R Y_u$$

$$-D_T \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2 \frac{\mathrm{d}\theta}{\mathrm{d}R}\right) = D\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\psi}{\mathrm{d}R}\right) = \frac{\psi}{\tau_{rb}}\mathrm{e}^{-\beta(1-\theta)}$$

$$R \leqslant R_f: \quad \theta = \theta_f, \quad \psi = 0; \qquad R \to \infty: \quad \theta = 0, \quad \psi = 1$$

## Asymptotic analysis $\beta \gg 1$

**Thin reaction zone** $\beta \to \infty$ thickness $\ll$ flame radius $R_f$

$$x \equiv R - R_f, \quad |x| \ll R_f: -D_T\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} = D\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} = \frac{\psi}{\tau_{rb}}\mathrm{e}^{-\beta(1-\theta)}$$

$$\eta \equiv \beta(x/R_f) = O(1), \qquad \eta \in [-\infty, +\infty] \qquad \psi \equiv \mathrm{Le}(\theta_f - \theta) \qquad \psi \mathrm{e}^{-\beta(1-\theta)} = \mathrm{e}^{\beta(\theta_f - 1)}(\mathrm{Le}/\beta)\Theta \mathrm{e}^{-\Theta}$$

$$\Theta \equiv \beta(\theta_f - \theta) = O(1) \qquad \Theta \in [0, \infty] \qquad \frac{\mathrm{d}^2\Theta}{\mathrm{d}\eta^2} = \frac{R_f^2}{D_T}\frac{\mathrm{e}^{\beta(\theta_f - 1)}}{\beta^2\tau_{rb}}\mathrm{Le}\,\Theta\mathrm{e}^{-\Theta}$$

Inner variables

$$\times\frac{\mathrm{d}\Theta}{\mathrm{d}\eta} \quad + \quad \int_0^\Theta \mathrm{d}\Theta \quad \Rightarrow \quad \beta \to \infty: \quad \boxed{-\lim_{\Theta\to\infty} D_T\frac{\mathrm{d}\theta}{\mathrm{d}R} = \mathrm{e}^{\beta(\theta_f - 1)/2}\sqrt{2\,\mathrm{Le}\frac{D_T}{\beta^2\tau_{rb}}}}$$

**External zones**

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\theta}{\mathrm{d}R}\right) = 0 \qquad R^2\frac{\mathrm{d}\theta}{\mathrm{d}R} = \text{cst.} \qquad R \geqslant R_f: \quad \boxed{\theta = \frac{1}{\mathrm{Le}}\frac{R_f}{R}}, \qquad R = R_f: \quad \boxed{D_T\frac{\mathrm{d}\theta}{\mathrm{d}R} = -\frac{1}{\mathrm{Le}}\frac{D_T}{R_f}} \qquad R < R_f: \theta = \theta_f$$

$$R = R_f: \quad \theta = \theta_f = 1/\mathrm{Le}$$

**Radius of the kernel**

matching $\Rightarrow \dfrac{1}{\mathrm{Le}}\dfrac{D_T}{R_f} = \mathrm{e}^{\frac{\beta}{2}\left(\frac{1}{\mathrm{Le}} - 1\right)}\sqrt{\mathrm{Le}\dfrac{D_T}{\tau_b}} \Leftrightarrow \boxed{\dfrac{R_f}{d_L} = \mathrm{Le}^{-3/2}\mathrm{e}^{\frac{\beta}{2}\left(1 - \frac{1}{\mathrm{Le}}\right)}},$

$$\tau_b \equiv \beta^2\tau_{rb}/2, \qquad \underset{(\mathrm{Le}=1)}{d_L} \equiv \sqrt{D_T\tau_b}$$

$$\boxed{\mathrm{Le} < 1: \ R_f \ll d_L \qquad \mathrm{Le} > 1: \ R_f \gg d_L}$$

## Quasi-isobaric ignition as a nucleation problem

$$\frac{\overline{R}_f}{d_L} = \frac{1}{\mathrm{Le}^{3/2}} \mathrm{e}^{\frac{\beta}{2}\left(1-\frac{1}{\mathrm{Le}}\right)}$$

$$\mathrm{Le} < 1: \ \overline{R}_f \ll d_L \qquad \mathrm{Le} > 1: \ \overline{R}_f \gg d_L$$

**Lean hydrocarbon mixtures** $\mathrm{Le} > 1$ **are difficult to ignite** $(R_f > d_L)$

$$D_{C_nH_m} < D_{O_2} \approx D_T, \quad \mathrm{Le} \approx D_{O_2}/D_{C_nH_m} > 1$$

**Instability ?** (adiabatic condition)

$$\theta = \theta_f R_f / R \qquad \theta_f = 1/\mathrm{Le} = \mathrm{cst.}$$ preheated zone at rest

$$\mathrm{d}\theta/\mathrm{d}R|_{R=R_f} = -\theta_f/R_f$$

heat flux towards the preheated zone

convection should be added to restore equilibrium

$R_f \nearrow$ $R_f > \overline{R}_f$ : |diffusion fluxes| $\propto 1/R$ $\searrow$ convective flux $\mathrm{d}R_f/\mathrm{d}t > 0$ $R_f \nearrow$

$\boxed{\text{positive feed back: amplification}}$

$R_f \searrow$ $R_f < \overline{R}_f$ : |diffusion fluxes| $\propto 1/R$ $\nearrow$ convective flux $\mathrm{d}R_f/\mathrm{d}t < 0$ $R_f \searrow$

**Stability analysis**

**Stabilization** in the presence of heat loss for $\mathrm{Le} < 1$

Buckmaster *at al. Combust. Flame* (1990) **79**, 381-392

**Flame balls in microgravity** lean hydrogen mixtures, diameter $= 2 - 15$ mm

Ronney P *et al. AIAA J.* (1998) 36, 1361-1368

**Ignition** by a constant energy source

Deshaies, B, Joulin G.. *Combust. Sci. Technol.* (1984) **37**, 99-116

# Lecture 7: Flame kernels and quasi-isobaric ignition

# Critical radius near the flammability limits

He Clavin, *Combust. Flame* (1993) Part I and II, 93, 391-420

chemical-kinetics data: $T^*$ crossover temperature $\theta^* \equiv (T^* - T_u)/(T_b - T_u)$
composition: $T_b$ associated with the heat release $T_b - T_u = q/c_p$
flame temperature of a spherical flame: $T_f$, $\theta_f \equiv (T_f - T_u)/(T_b - T_u) = 1/\mathrm{Le}$

model of lecture VI

$$-D_T \frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\theta}{\mathrm{d}R}\right) = D\frac{1}{R^2}\frac{\mathrm{d}}{\mathrm{d}R}\left(R^2\frac{\mathrm{d}\psi}{\mathrm{d}R}\right) = \dot{W}$$

$$R \leqslant R_f: \quad \theta = \theta_f, \quad \psi = 0; \qquad R \to \infty: \quad \theta = 0, \quad \psi = 1$$

$$\begin{cases} T > T^*: & \dot{W} = \dfrac{\rho_b}{\rho_u}\dfrac{\psi^2}{\tau_{rb}}[\mathrm{e}^{-\beta(1-\theta)} - \mathrm{e}^{-\varepsilon}] \\ T < T^*: & \dot{W} = 0 \end{cases}$$

$$\frac{1}{\tau_{rb}} \equiv \frac{1}{\tau^*}\mathrm{e}^{\frac{E}{k_B}\left(\frac{1}{T^*}-\frac{1}{T_b}\right)} = \frac{B_{4f}}{B_{5f}}B_{1f}\mathrm{e}^{-\frac{E}{k_B T_b}}c^*_{O_2 u}$$

$$\varepsilon \equiv \beta\frac{(T_b - T^*)}{(T_b - T_u)} > 0 \qquad \beta \equiv \frac{E}{k_B T_b}\left(1 - \frac{T_u}{T_b}\right) \gg 1 \qquad \begin{cases} \varepsilon = O(1): \text{ near to flammability limits } (\varepsilon = 0: \text{ quenching } \dot{W} = 0) \\ \varepsilon \gg 1, \ \mathrm{e}^{-\varepsilon} \approx 0: \text{ far from flammability limits} \end{cases}$$

**Thin reaction zone** $\beta \to \infty$ (non-dimensional form $\zeta = x/d_L$, $d_L$ for $\varepsilon \gg 1$, $\mathrm{Le} = 1$, $2^{nd}$reaction order)

$$-\mathrm{d}^2\theta/\mathrm{d}\zeta^2 = \dot{\mathrm{w}}(\theta,\psi), \quad (1/\mathrm{Le})\mathrm{d}^2\psi/\mathrm{d}\zeta^2 = \dot{\mathrm{w}}(\theta,\psi) \qquad \theta = \theta_f : \psi = 0$$

$$\boxed{\frac{D_T}{d_L^2} = 4\frac{\rho_b}{\rho_u}\frac{1}{\beta^3\tau_{rb}}}$$

$$\psi = \mathrm{Le}(\theta_f - \theta)$$

$$\begin{cases} T > T^*: & \dot{\mathrm{w}} = (\beta^3/4)\mathrm{Le}^2\mathrm{e}^{\beta(\theta_f - 1)}\left\{(\theta_f - \theta)^2\left[\mathrm{e}^{\beta(\theta - \theta_f)} - \mathrm{e}^{-\varepsilon - \beta(\theta_f - 1)}\right]\right\} \\ T < T^*: & \dot{\mathrm{w}} = 0 \end{cases}$$

$$\theta^* \leqslant \theta < \theta_f$$

$$\theta^* \equiv \frac{T^* - T_u}{T_b - T_u} = 1 - \frac{\varepsilon}{\beta}$$

$$\times\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \Rightarrow \left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 = \frac{\mathrm{Le}^2}{2}\mathrm{e}^{\beta(\theta_f - 1)}\int_{\substack{\theta^* = 1 - \varepsilon/\beta \\ \dot{\mathrm{w}} = 0}}^{\theta_f} \beta^3(\theta_f - \theta)^2\left\{\mathrm{e}^{\beta(\theta - \theta_f)} - \mathrm{e}^{-\varepsilon - \beta(\theta_f - 1)}\right\}\mathrm{d}\theta$$

at the exit of the reaction layer
(entrance of the preheated zone)

$$\left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 = \frac{\mathrm{Le}^2}{2}\mathrm{e}^{\beta(\theta_f-1)} \int\limits_{\theta^*=1-\varepsilon/\beta}^{\theta_f} \beta^3(\theta_f-\theta)^2\left\{\mathrm{e}^{\beta(\theta-\theta_f)} - \mathrm{e}^{-\varepsilon-\beta(\theta_f-1)}\right\}\mathrm{d}\theta$$ at the exit of the reaction layer (entrance of the preheated zone)

$$\Theta = \beta(\theta_f - \theta) \in [0, \Theta_f] \qquad \Theta_f \equiv \beta\left(\frac{T_f - T^*}{T_b - T_u}\right) = \beta(\theta_f - \theta^*) = \varepsilon + \beta(\theta_f - 1) \geqslant 0$$

$$\mathrm{d}\theta = -\beta \mathrm{d}\Theta$$

measure of the distance from the flammability limit: $\Theta_f \in [0, \infty]$

$$(\mathrm{d}\theta/\mathrm{d}\zeta)^2 = \mathrm{Le}^2\mathrm{e}^{\beta(\theta_f-1)}J(\Theta_f) \qquad J(\Theta_f) \equiv \frac{1}{2}\int_0^{\Theta_f}\Theta^2(\mathrm{e}^{-\Theta} - \mathrm{e}^{-\Theta_f})\mathrm{d}\Theta \in [0,1]$$

$$J(\Theta_f) = 1 - \mathrm{e}^{-\Theta_f}\left(1 + \Theta_f + \frac{\Theta_f^2}{2!} + \frac{\Theta_f^3}{3!}\right) \quad \begin{cases} \Theta_f \gg 1: & J \approx 1 \\ 0 \leqslant \Theta_f \ll 1: & J \approx \Theta_f^4/(4!) \end{cases}$$

## Preheated zone and matching

flame temperature of the spherical flame

$$R \geqslant R_f: \quad \frac{\mathrm{d}\theta}{\mathrm{dR}} = -\theta_f\frac{R_f}{R^2}, \quad \boxed{\theta_f = \frac{1}{\mathrm{Le}}}$$

$d_L$ for $\varepsilon \gg 1,\ \mathrm{Le} = 1,\ 2^{nd}$reaction order

$$\beta \to \infty: \quad \boxed{\frac{d_L}{R_f} = \mathrm{Le}^2\mathrm{e}^{\frac{\beta}{2}\left(\frac{1}{\mathrm{Le}}-1\right)}\sqrt{J(\Theta_f)}}$$

$$T_f \to T^* \Rightarrow R_f/d_L \to \infty$$

$$\Theta_f \to 0$$

$T_b$ is determined by the composition of the mixture $Y_{Ru}$ (mass fraction of the limiting component)

$T^*$ is determined by the chemical kinetics $Y_{Ru}$

$$\theta \equiv \frac{T - T_u}{T_b - T_u}$$

$\boxed{\theta^* \text{ depends on the composition}}$

$$\theta^* \equiv \frac{T^* - T_u}{T_b - T_u}$$

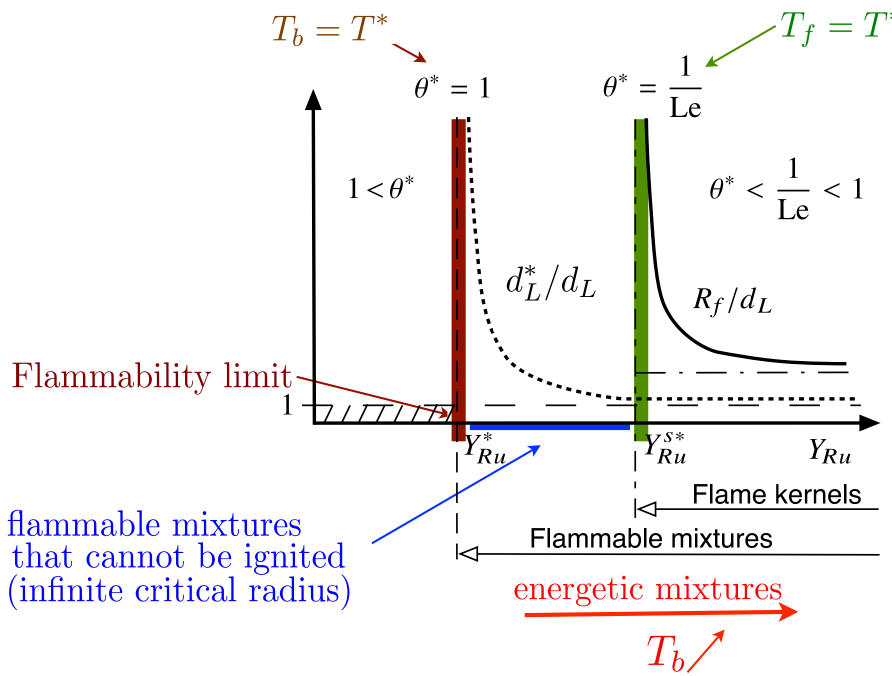
temperature of the planar flame depends on the composition

temperature of the spherical flame kernel

$$\theta_f \equiv \frac{T_f - T_u}{T_b - T_u} = \frac{1}{\mathrm{Le}}$$

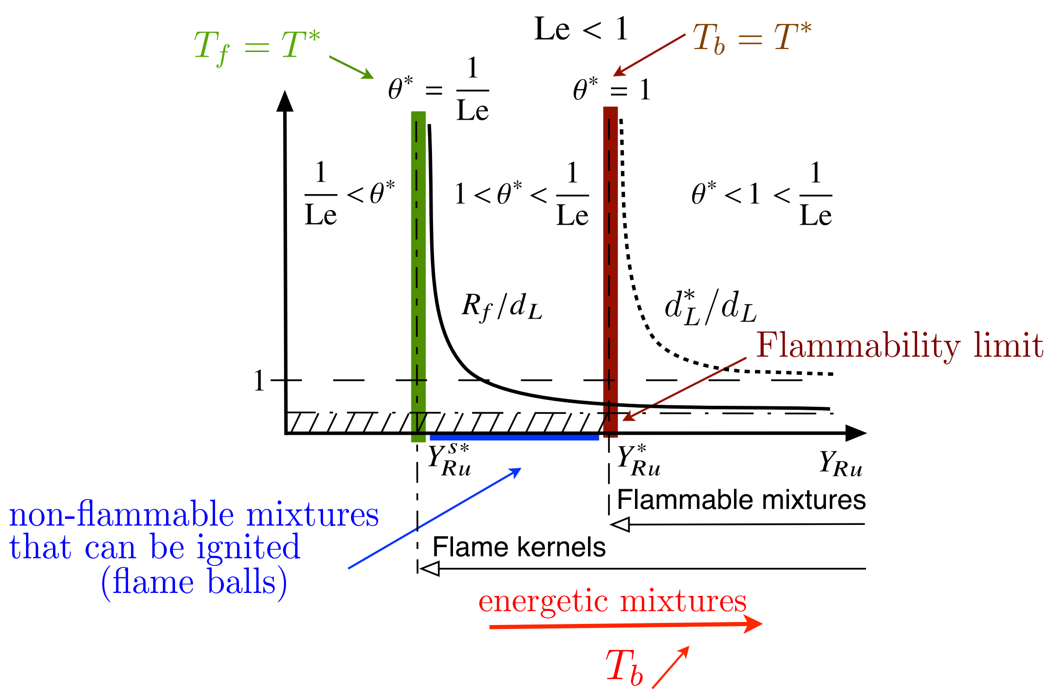
$\boxed{\theta_f = 1/\mathrm{Le}}$

$\mathrm{Le} > 1 \Rightarrow T_f < T_b$




Le $> 1$ : Heavy hydrocarbon lean mixtures
Hydrogen rich mixtures

$\mathrm{Le} < 1 \Rightarrow T_f > T_b$



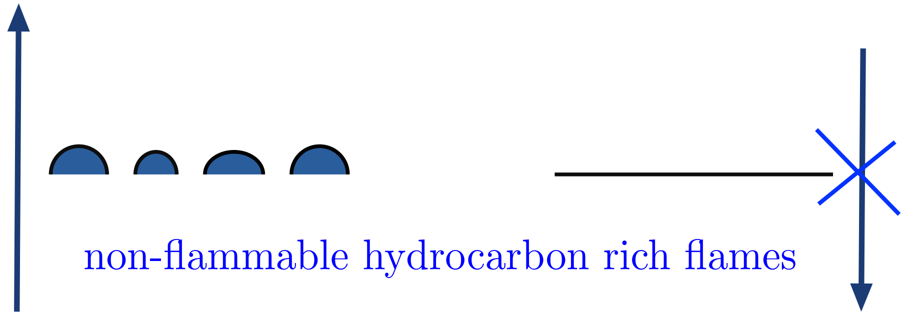

Le $< 1$ : Heavy hydrocarbon rich mixtures
Hydrogen lean mixtures

He Clavin, *Combust. Flame* (1993) Part I and II, 93, 391-420

## Flammability limits Critical conditions of ignition

Some hydrocarbon lean mixtures that are flammable cannot be ignited quasi-isobarically
Some hydrocarbon rich mixtures that are non-flammable can sustain curved flames

## Limits for upstream propagation $\neq$ downstream propagation




Buoyancy promoted curved flames

## Ignition in turbulent flows

Wu, F. *et al.* *Phys. Rev. Lett.* (2014) 113, 024503

Turbulence facilitates ignition of hydrocarbon lean mixtures
Turbulence may suppress ignition of hydrocarbon rich mixtures

simplest explanation:

Turbulent diffusion coefficients are all equal $\Leftrightarrow$ $\left\{ \begin{array}{l} \mathrm{Le} > 1 \\ \mathrm{Le} < 1 \end{array} \right. \longrightarrow \mathrm{Le} = 1$

**laminar** **turbulent**

# Lecture 7: Flame kernels and quasi-isobaric ignition

# Dynamics of slowly expanding flame kernels

## Quasi-steady preheated zone of flame kernel ?

preheated zone in the reference frame attached to $R_f(t)$ $\quad \dot{R}_f \equiv \dfrac{\mathrm{d}R_f}{\mathrm{d}t}$

$$\cancel{\frac{\partial \theta}{\partial t}} - \dot{R}_f \frac{\partial \theta}{\partial R} - D_T \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 \frac{\partial \theta}{\partial R}\right) = 0$$

quasi-steady state ? $\boxed{t_{relax} \equiv \dfrac{R^2}{D_T} \ll t_{evol} \equiv \dfrac{R_f}{\dot{R}_f}}$ $R \ll \sqrt{D_T t_{evol}}$, not valid at **large distance**

## The evolution of spherical flame kernel cannot be quasi-steady at large distance

exact solution of the heat equation with a point energy source $\partial T/\partial t = D_T \Delta T$ $\quad$ point source, $R = 0$, $t > 0$ : $\dot{Q}(t)$

$$T(R,t) - T_u = \int_0^t \frac{\dot{Q}(t-\tau)}{\rho c_p} \frac{\exp(-R^2/4D_T\tau)}{(4\pi D_T \tau)^{3/2}} \mathrm{d}\tau$$

$\dot{Q} = \text{cst.}$ $\qquad X' \equiv R/\sqrt{4D_T\tau} \qquad \mathrm{d}X' = -2D_T R \dfrac{\mathrm{d}\tau}{(4D_T\tau)^{3/2}}$

$$T - T_u = \frac{1}{4\pi D_T} \frac{\dot{Q}}{\rho c_p} \frac{1}{R} \frac{2}{\sqrt{\pi}} \int_{R/\sqrt{4D_T t}}^{\infty} \mathrm{d}\,\mathrm{X}' \mathrm{e}^{-X'^2}$$

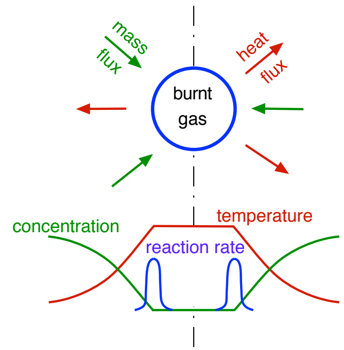
relax time toward $(T - T_u) \propto 1/R$ increases with $R$ like $R^2/D_T$

$R^2/(4D_T t) \to 0$

For $\boxed{\mathrm{Le} < 1}$ and near to **the Zeldovich radius** the **slow** evolution of flame kernels is governed by the diffusion at **large** distance

Zeldovich radius $\quad R_{fZ}/d_L = \mathrm{e}^{\beta(\mathrm{Le}-1)/2\mathrm{Le}}$

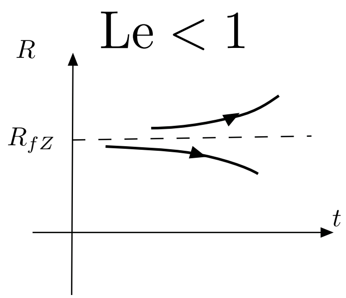
$$\tau \equiv t/t_{ref} \qquad \sqrt{t_{ref}} \equiv \frac{\beta(1-\mathrm{Le}^{1/2})}{\mathrm{Le}} \frac{R_{fZ}}{(4\pi D_T)^{1/2}} \qquad \mathrm{r}_f \equiv R_f/R_{fZ}$$




Joulin's equation (Joulin 1985)

$$\frac{\beta}{2}\left(\theta_f - \frac{1}{\mathrm{Le}}\right) = -\int_0^{\tau} \frac{\mathrm{d}\tau'}{\tau'^{1/2}} \dot{\mathrm{r}}_f(\tau-\tau')$$

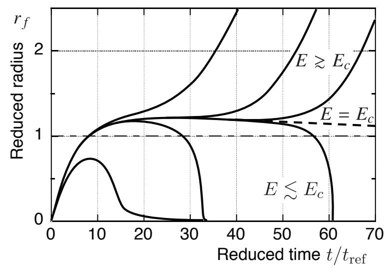
$$\frac{1}{\mathrm{r}_f} = \exp\left[-\int_0^{\tau} \frac{\mathrm{d}\tau'}{\tau'^{1/2}} \dot{\mathrm{r}}_f(\tau-\tau')\right]$$




**The structure and the dynamics of flame kernels** $\neq$ planar flames even for $R_f \gg d_L$ $(\theta \approx \theta_f/R)$

Extension to a short pulse of an energy source

Joulin G.. *Combust. Sci. Technol.* (1985) **43**, 99-113




Extension to the proximity of flammability limits + heat loss

Clavin P. *Combust. Flame* (2017) **175**, 80-90

Dynamical quenching of flame kernels in **nonflammable mixtures** for $\mathrm{Le} < 1$

$$\frac{1}{\sqrt{\mathrm{r}_f}} + H_b \mathrm{r}_f^2 = 1 - I(\tau) \quad \text{where} \quad I(\tau) \equiv \int_0^{\tau} \frac{\mathrm{d}\tau'}{\tau'^{1/2}} \dot{\mathrm{r}}_f(\tau-\tau')$$

Self-extinguished flames in micro-gravity experiments of lean methane-air mixtures

Ronney P. *Combust. Flame* (1985) **62**, 121-133
Ronney P. *Combust. Flame* (1990) 82, 1-14
Ronney P *et al.* *AIAA J.* (1998) 36, 1361-1368

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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ECM & CNRS

## Lecture VIII
## Thermo-acoustic instabilities. Vibratory flames

# Lecture 8: Thermo-acoustic instabilities

## Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*
*Sound emission by a localized heat source*
*Linear growth rate*

## Lecture 8-2. Admittance & transfer function

*Flame propagating in a tube*
*Pressure coupling*
*Velocity and acceleration coupling*

## Lecture 8-3. Vibratory instability of flames

*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*
*Flame propagating downward (sensitivity to the Markestein number)*
*Bunsen flame in an acoustic field*

# VIII-1) Rayleigh criterion

## Acoustic waves in a reactive medium

Lord Rayleigh 1878

Ideal gas

$$p = (c_p - c_v)\rho T \qquad \frac{c_p}{c_p - c_v}\frac{\mathrm{D}p}{\mathrm{D}t} = c_p T \frac{\mathrm{D}\rho}{\mathrm{D}t} + c_p \rho \frac{\mathrm{D}T}{\mathrm{D}t} \qquad \mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \mathbf{u}.\nabla$$

$$a^2 = (c_p/c_v)(c_p - c_v)T \qquad \rho c_p \frac{\mathrm{D}}{\mathrm{D}t}T = \frac{c_p}{c_p - c_v}\frac{\mathrm{D}}{\mathrm{D}t}p - \frac{c_v}{c_p - c_v}a^2\frac{\mathrm{D}}{\mathrm{D}t}\rho$$

Energy conservation

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + \nabla.(\lambda \nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}$$

elimination of D T/Dt

$$\frac{c_v}{c_p - c_v}\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{c_v}{c_p - c_v}a^2\frac{\mathrm{D}\rho}{\mathrm{D}t} = \nabla.(\lambda \nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}$$

$$\boxed{\mathrm{D}p/\mathrm{D}t - a^2\mathrm{D}\rho/\mathrm{D}t = \dot{q}_\gamma}$$

isentropic acoustic
$\delta p = a^2 \delta\rho$

$$\dot{q}_\gamma \equiv (\gamma - 1)\left[\nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}\right] \qquad \gamma \equiv c_p/c_v$$

$$\dot{q}_\gamma(\mathbf{r}, t) = \text{ heat transfer + heat release}$$

(rate of energy transfert per unit volume)

$$\boxed{\mathrm{D}p/\mathrm{D}t - a^2\mathrm{D}\rho/\mathrm{D}t = \dot{q}_\gamma}$$

$$\dot{q}_\gamma \equiv (\gamma - 1)\left[\nabla.(\lambda\nabla T) + \sum_j Q^{(j)}\dot{W}^{(j)}\right] \qquad \gamma \equiv c_p/c_v$$

Linearization around a uniform state $\nabla\overline{a} \approx 0$,

$$\rho = \overline{\rho} + \rho', \quad p = \overline{p} + p', \quad \mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}', \quad \dot{q}_\gamma = \overline{\dot{q}_\gamma} + \dot{q}'_\gamma$$

Mean flow velocity neglected in front of the sound speed $\overline{\mathbf{u}}.\nabla \approx 0$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \;\; \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p \quad \Rightarrow \quad \partial\rho'/\partial t = -\overline{\rho}\nabla.\mathbf{u}', \;\; \overline{\rho}\partial\mathbf{u}'/\partial t = -\nabla p',$$

$$\partial^2\rho'/\partial t^2 = \Delta p'$$

Approximations

$a, \; c_p, \; c_v;$ constant

$\partial/\partial t$ $$\partial p'/\partial t - \overline{a}^2\partial\rho'/\partial t = \dot{q}'_\gamma$$

elimination of $\rho'$ $$\boxed{\partial^2 p'/\partial t^2 - \overline{a}^2\Delta p' = \partial\dot{q}'_\gamma/\partial t}$$

## Sound emission by a localized heat source in free space

classical problem of acoustics

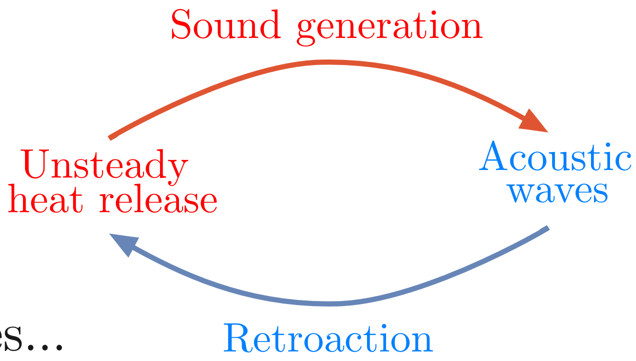
acoustic wavelength $\gg$ size of the combustion zone $\partial\dot{q}'_\gamma(\mathbf{r},t)/\partial t = \delta(\mathbf{r})\ddot{\Omega}(t)$ $\quad \ddot{\Omega}(t) \equiv \partial\dot{\Omega}(t)/\partial t,$ $\boxed{\dot{\Omega}(t) = \iiint \dot{q}'_\gamma(\mathbf{r}',t)\mathrm{d}^3\mathbf{r}'}$

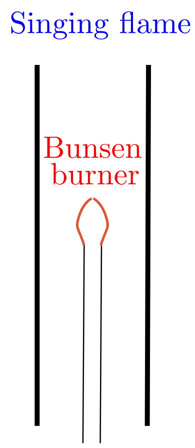
Green's retarded propagator $\quad (1/\overline{a}^2)\partial^2 G/\partial t^2 - \Delta G = \delta(\mathbf{r})\delta(t), \quad G(\mathbf{r},t) = \overline{a}\delta(r - \overline{a}t)/4\pi r$

spherical geometry $$\boxed{p'(\mathbf{r},t) =}\frac{1}{4\pi\overline{a}^2}\iiint\frac{1}{r'}\frac{\partial}{\partial t}\dot{q}'_\gamma(\mathbf{r}',t - r/\overline{a})\mathrm{d}^3\mathbf{r}' = \boxed{\frac{\ddot{\Omega}(t - r/\overline{a})}{4\pi\overline{a}^2 r}}, \qquad r = |\mathbf{r}|$$

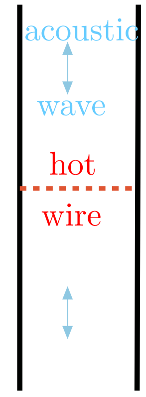
# Liner growth rate

retro-action loop:




Higgings 1802




Rijike 1859

Rocket engines, gas turbines...

Simplest retro-action mechanism: pressure coupling + 1-D geometry

$$\boxed{\delta \dot{q}_\gamma = b\,\delta p/\tau_{ins}} \qquad \delta p(x,t) = \sum_{k=-\infty}^{\infty} \tilde{p}_k(t)\mathrm{e}^{\mathrm{i}kx} \qquad k = 2\pi n/L$$

$$\partial^2 p'/\partial t^2 - \bar{a}^2 \Delta p' = \partial \dot{q}'_\gamma/\partial t \qquad \Rightarrow \frac{\mathrm{d}^2\tilde{p}_k}{\mathrm{d}t^2} - \frac{b}{\tau_{ins}}\frac{\mathrm{d}\tilde{p}_k}{\mathrm{d}t} + \bar{a}^2k^2\tilde{p}_k = 0$$

$$\tilde{p}_k(t) \propto \mathrm{e}^{\sigma(k)t} \qquad 2\sigma\tau_{ins} = b \pm \sqrt{\cancel{b^2} - 4\bar{a}^2k^2\tau_{ins}^2} \qquad \frac{1}{\tau_{ins}} \ll \omega_k = \bar{a}k$$

$$\mathrm{Im}(\sigma) = \omega_k + .., \quad \boxed{\mathrm{Re}(\sigma) = b/(2\tau_{inst}) + ..} \Rightarrow \begin{cases} b > 0 \text{ fluctuations of heat release and pressure } \boxed{\text{in phase: instability}} \\ b < 0 \text{ fluctuations of heat release and pressure } \boxed{\text{out of phase: stability}} \end{cases}$$

More general retro-action mechanism $\boxed{\delta \dot{q}'_\gamma(x,t) = \dfrac{1}{\tau_{ins}}\displaystyle\int_{-\infty}^{t} b(t-t')\delta p'(x,t')\mathrm{t}'}$

$$b(\tau) = \int_{-\infty}^{+\infty} r(\omega)\mathrm{e}^{\mathrm{i}\omega\tau_d(\omega)}\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\omega + \mathrm{c.c.} \qquad r(\omega) > 0 \qquad \omega\tau_d(\omega) \text{ is the phase lag}$$

$$\boxed{-\pi/2 < \omega_k\tau_d(\omega_k) < +\pi/2 \;:\; \text{Instability}}$$

Nonlinear study: limit cycles in the unstable case

# Lecture 8: Thermo-acoustic instabilities

## Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*
*Sound emission by a localized heat source*
*Linear growth rate*

## Lecture 8-2. Admittance & transfer function

*Flame propagating in a tube*
*Pressure coupling*
*Velocity and acceleration coupling*

## Lecture 8-3. Vibratory instability of flames

*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*
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*Bunsen flame in an acoustic field*

# Admittance & transfer function

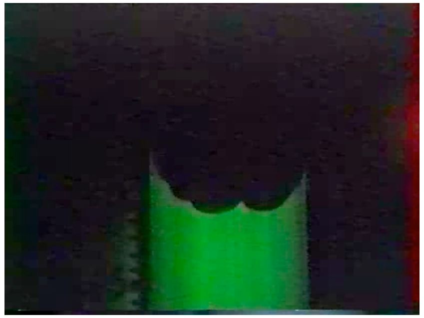
## Flame propagating in a tube

thickness of the flame brush $\ll$ acoustic wavelength

gas expansion $\Rightarrow$ jump of the fluctuations of the flow velocity (acoustics)

$$(\delta u_b - \delta u_u)/U_L = O(1)$$

$x$



burned gas

$\uparrow\ \delta u_b$

$\uparrow\ \delta u_u$

fresh mixture

Tomography laser

Boyer, L. *Combust. Flame* (1980) **39**, 321-323

jump of pressure across the flame brush is negligible

acoustic pressure

$$\delta p = \rho a \delta u \qquad (\delta p_b - \delta p_u)/p = O(U_L/a)$$

$$p/\rho a^2 = O(1)$$

$\delta p_f$ : fluctuation of the pressure at the flame

averaged (per period) flux of combustion energy transferred to the acoustic waver

$$\boxed{\dot{\mathcal{E}}_t = \overline{(\delta u_b - \delta u_u)\delta p_f}}$$

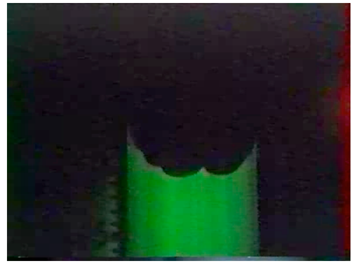
mass conservation (quasi-isobaric combustion) $\quad \nabla.\mathbf{u} = \dfrac{1}{T}\dfrac{\mathrm{D}T}{\mathrm{D}t} = \dfrac{\dot{q}_\gamma/(\gamma-1)}{\rho c_p T} = \dfrac{\dot{q}_\gamma}{\rho a^2}$

$$\boxed{(\delta u_b - \delta u_u) = \int_{\text{flame brush}} \frac{\delta \dot{q}_\gamma}{\rho a^2}\mathrm{d}x}$$

$$\boxed{\dot{\mathcal{E}}_t = \overline{(\delta u_b - \delta u_u)\delta p_f}}$$

$$(\delta u_b - \delta u_u) = \int_{\text{flame brush}} \frac{\delta \dot{q}_\gamma}{\rho a^2} \mathrm{d}x$$

## Pressure coupling




laser tomography (Boyer 1980)

### Definition of the admittance function $\mathcal{Z}(\omega)$

$$\delta u(t) = \mathrm{Re}\left[\hat{u}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right] \qquad \delta p(t) = \mathrm{Re}\left[\hat{p}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right]$$

$$\boxed{(\hat{u}_b - \hat{u}_u) = \mathcal{Z}(\omega)\, \hat{p}_f / \rho_b a_b} \quad \Leftarrow \quad (\text{Rayleigh: } \delta \dot{q}_\gamma \text{ v.s. } \delta p)$$

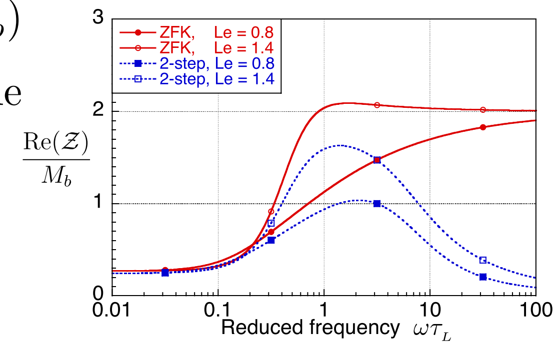
$$\dot{\mathcal{E}}_t = \frac{1}{4\rho_b a_b}(\mathcal{Z}\hat{p}_f\hat{p}_f^* + \mathcal{Z}^*\hat{p}_f\hat{p}_f^*) = \frac{1}{2}[\mathrm{Re}\,\mathcal{Z}(\omega)]\frac{|\hat{p}_f|^2}{\rho_b a_b}$$

$$\boxed{\text{instability : } \quad \mathrm{Re}(\mathcal{Z}) > 0}$$

### Analytical study of a planar flame submitted to a fluctuation of pressure $(\beta \to \infty)$ $\quad \delta T_f/T_f \propto \delta p_f/p_f$

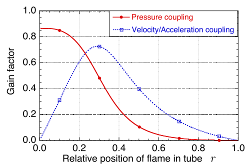
$|\mathcal{Z}| = \mathrm{O}(M_b)$

gaseous flame




$$\frac{\tau_a}{\tau_{ins}} \propto (\gamma - 1)M_b \frac{E}{k_B T_b}$$

coeff depends on
the position in the tube
as $\delta p_f$ does



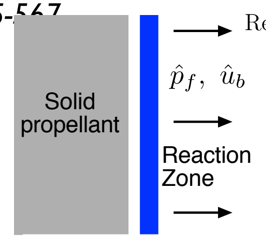

weak coupling

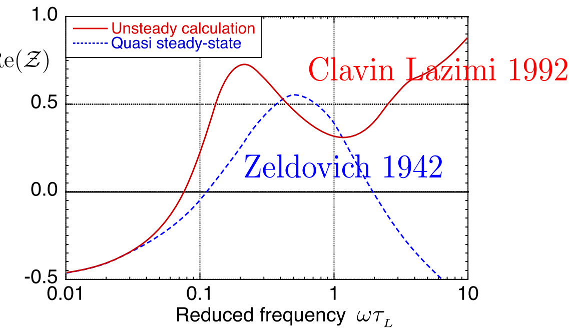
$$\tau_a/\tau_{ins} \ll 1$$

P. Clavin et al. (1990), *J. Fluid Mech.* ***216,*** 299-322

P. Clavin & G.Searby (2008), *Combust. Theor. Model.* ***12*** *(3), 545-567*

## Solid propellant



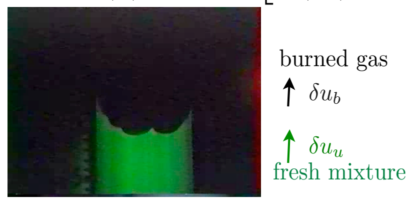




P. Clavin & D. Lazimi (1992), *Combust. Sci. Technol* ***83***, *1-32*

J. Garcia-Schafer & A. Linan (2001), J. Fluid Mech. ***437***, *229-254*

$\delta u(t) = \mathrm{Re}\left[\hat{u}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right]$ $\delta p(t) = \mathrm{Re}\left[\hat{p}(\omega)\mathrm{e}^{\mathrm{i}\omega t}\right]$

# Velocity and acceleration coupling




fluctuating velocity $\Rightarrow$ modification to flame geometry

$\Rightarrow$ fluctuation of heat release through the flame surface

**Transfer function for a flame in a tube** $\mathcal{T}_r(\omega)$ $\boxed{(\hat{u}_b - \hat{u}_u) = \mathcal{T}_r(\omega)\hat{u}_u}$ $\dot{\mathcal{E}}_t = (1/4)(\mathcal{T}_r\hat{u}_u\hat{p}_f^* + \mathcal{T}_r^*\hat{u}_u^*\hat{p}_f)$

$\hat{u}\hat{p}_f^* = -\hat{u}^*\hat{p}_f$ $\dot{\mathcal{E}}_t = \mathrm{Im}\,\mathcal{T}_r(\omega)(\mathrm{i}\hat{u}_u\hat{p}_f^*)/2$

phase quadrature (acoustic mode of a tube) Real number (sign depends on position)

## Weakly cellular flame propagating downward in an acoustic wave

acceleration of a curved flame $\Rightarrow$ modulation of the flame surface $S = \int \mathrm{d}y\sqrt{1+\alpha_y'^2}$

$\int \delta\dot{q}\,\mathrm{d}x = \rho_u U_L c_p (T_b - T_u)\,\delta S/S_o$ $\delta u_b - \delta u_u = \int \frac{\delta\dot{q}}{\rho a^2}\mathrm{d}x$ $\Rightarrow$ $\boxed{\delta u_b - \delta u_u = (T_b/T_u - 1)U_L\delta S/S_o}$

fluctuation of heat release rate/ cross-section aera

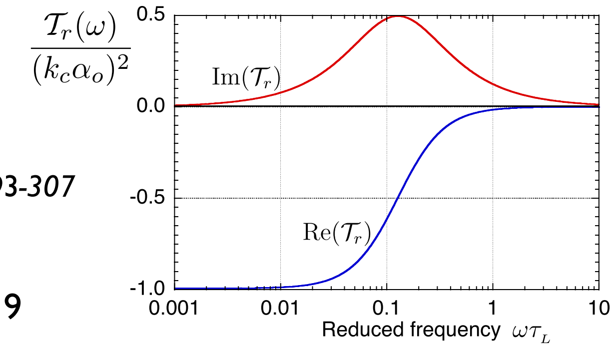
Consider a curved front slightly perturbed

$x = \alpha(y,t)$ $\alpha(y,t) = \tilde{\alpha}(t)\cos(ky)$ $\tilde{\alpha}(t) = \underbrace{\tilde{\alpha}_0}_{\text{unperturbed}} + \underbrace{\hat{\alpha}_1\mathrm{e}^{\mathrm{i}\omega t}}_{\text{perturbation}} + c.c$

$k\tilde{\alpha}_0 \ll 1$ $|\tilde{\alpha}_1| \ll \tilde{\alpha}_0$ (linear response ok) $\Rightarrow$ $\boxed{\delta S/S_o = (k^2/2)\tilde{\alpha}_o\hat{\alpha}_1\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{c.c.}}$

$\tilde{\alpha}_1$ vs $\hat{u}_u$ ? $g'(t) = \mathrm{Re}\left[i\omega\,\hat{u}_u\mathrm{e}^{\mathrm{i}\omega t}\right]$ $\bar{g} > 0$

lecture IV: $\left(1+\frac{\rho_b}{\rho_u}\right)\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}t^2} + 2(U_L k)\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}t} - \left(\frac{\rho_u}{\rho_b} - 1\right)k\left[-\frac{\rho_b}{\rho_u}[\bar{g} + g'(t)] + U_L^2 k\left(1 - \frac{k}{k_m}\right)\right]\tilde{\alpha} = 0$




Analytical expression ($k = k_c$)

P. Pelcé and D. Rochwerger (1992), *J. Fluid Mech.* **239,** 293-307

**ok for the primary instability**

# Lecture 8: Thermo-acoustic instabilities

## Lecture 8-1. Rayleigh criterion

*Acoustic waves in a reactive medium*
*Sound emission by a localized heat source*
*Linear growth rate*

## Lecture 8-2. Admittance & transfer function

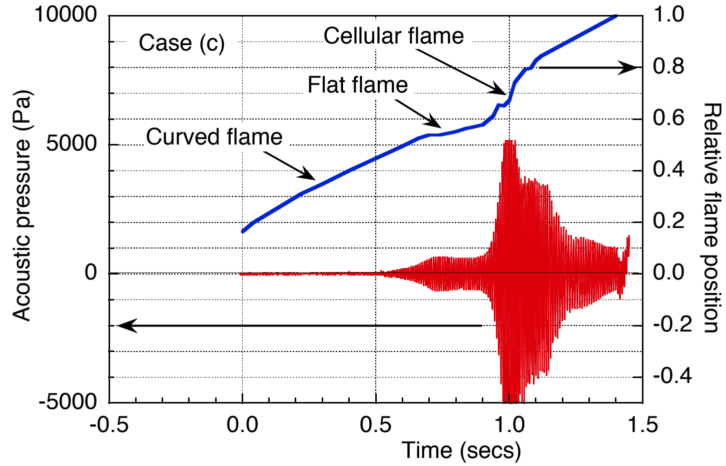
*Flame propagating in a tube*
*Pressure coupling*
*Velocity and acceleration coupling*

## Lecture 8-3. Vibratory instability of flames

*Acoustic re-stabilisation and parametric instability (Mathieu's equation)*
*Flame propagating downward (sensitivity to the Markestein number)*
*Bunsen flame in an acoustic field*

# VIII-3) Vibratory instability of flames

primary instability + re-stabilisation + parametric instability


## Acoustic re-stabilisation and parametric instability

Markstein 1964

$$\tau' \equiv t/\tau_h, \quad \tau_h \equiv 1/(U_L k), \quad \varpi \equiv \omega\tau_h, = (\omega\tau_L)/(kd_L) \quad \kappa = kd_L$$

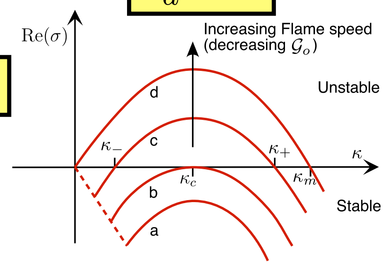
$$\boxed{u_a = 0}$$

$$\boxed{g'(t) = \omega u_a U_L \cos(\omega t)}$$

$$\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}\tau'^2} + 2\mathsf{B}\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}\tau'} + \left[-\mathsf{D} + \varpi^2\mathsf{C}\cos(\varpi\tau')\right]\tilde{\alpha} = 0$$

$$\upsilon_b \equiv \rho_u/\rho_b > 1 \qquad \mathsf{B} \equiv \frac{\upsilon_b}{\upsilon_b + 1} \qquad \mathsf{C} \equiv \left(\frac{\upsilon_b - 1}{\upsilon_b + 1}\right)\frac{u_a}{\varpi},$$

$$\mathsf{D} \equiv \upsilon_b\left(\frac{\upsilon_b - 1}{\upsilon_b + 1}\right)\frac{\mathsf{N}}{\kappa} \qquad \mathsf{N}(\kappa) \equiv -\mathcal{G}_o + \kappa - \kappa^2/\kappa_m \qquad \mathcal{G}_o \equiv \upsilon_b^{-1}|g|d_L/U_L^2$$

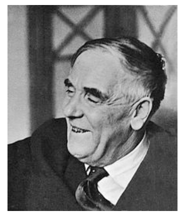



## Mathieu's equation. Kapitza pendulum

$$\mathrm{t} \equiv \varpi\tau' \qquad Y(\mathrm{t}) \equiv \mathrm{e}^{\mathsf{B}\tau'}\tilde{\alpha}$$

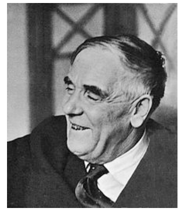
$$\boxed{\frac{\mathrm{d}^2 Y}{\mathrm{dt}^2} + \{\Omega + h\,\cos(\mathrm{t})\}\,Y = 0}$$

$$\Omega = -\frac{(\mathsf{D} + \mathsf{B}^2)}{\varpi^2} \qquad h = \mathsf{C}$$


Kapitza 1951

## Mathieu's equation. Kapitza pendulum

Faraday 1831

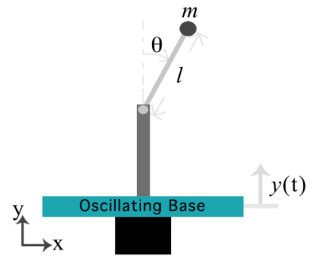


Kapitza 1951

$$\Omega = \text{constant} \qquad \boxed{\frac{\mathrm{d}^2 Y}{\mathrm{dt}^2} + \{\Omega + h\,\cos(\mathrm{t})\}\,Y = 0}$$

$\Omega > 0$ : Oscillator whose frequency $\sqrt{\Omega}$ is modulated   Prametric instability (Faraday 1831)

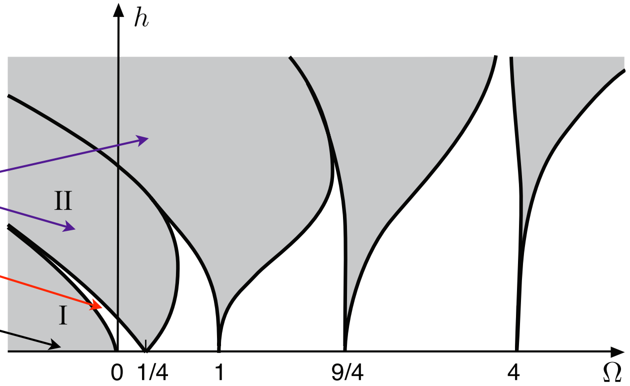
$\Omega < 0$ : Re-stabilization of an unstable position of a pendulum by oscillations (Kapitza 1951)




Parametric instability

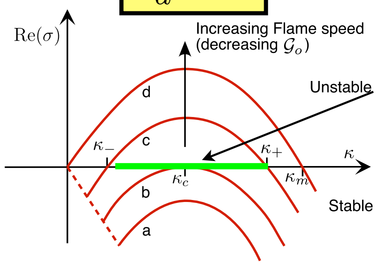
Re-stabilization tongue

Unstable situation




Stability limits of the solutions to Mathieu's equation
White regions: stable. Grey regions unstable

$u_a = 0$




unstable wavelengths

### Flame propagating downward

$g'(t) = \omega u_a U_L \cos(\omega t)$

$$\frac{\mathrm{d}^2\tilde{\alpha}}{\mathrm{d}\tau'^2} + 2\mathsf{B}\frac{\mathrm{d}\tilde{\alpha}}{\mathrm{d}\tau'} + \left[-\mathsf{D} + \varpi^2\mathsf{C}\cos(\varpi\tau')\right]\tilde{\alpha} = 0$$

**G.H. Markestein** (1964) *Nonsteady flame propagation New York: Pergamon*

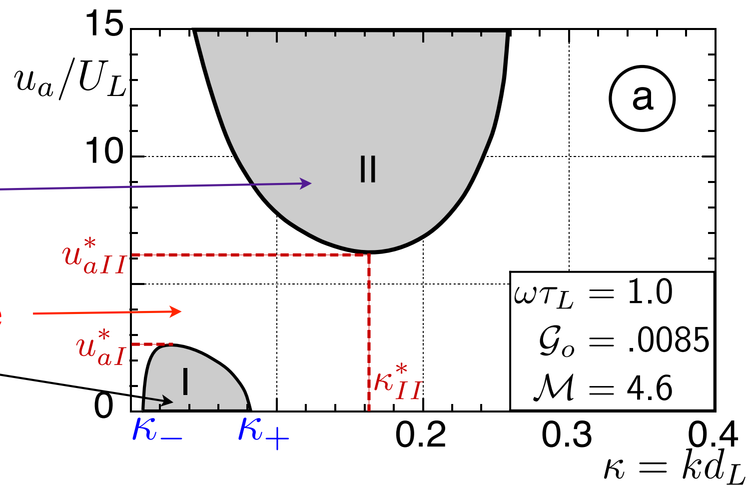
Parametric instability

$$u_{aI}^{*2} \approx 2\upsilon_b \frac{(\upsilon_b+1)}{(\upsilon_b-1)}\left(1 - \frac{U_{Lc}}{U_L}\right), \qquad \frac{k_I^*}{k_m} \approx \frac{1}{2}\frac{U_{Lc}}{U_L}$$

$$u_{aII}^* = 2\upsilon_b/(\upsilon_b - 1)$$

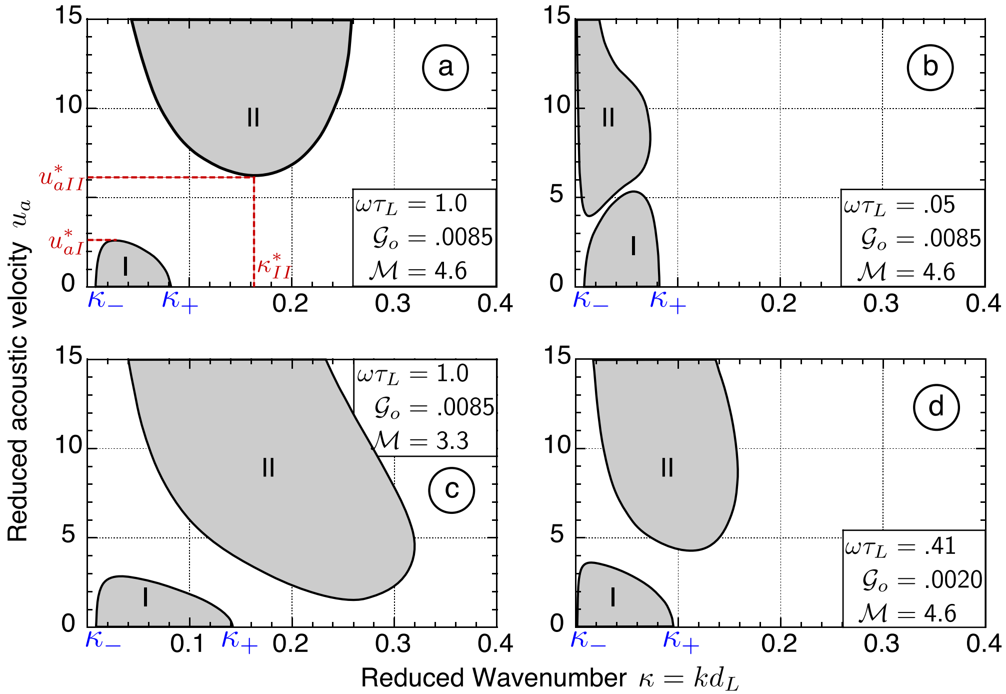
$u_{aI}^* < u_a < u_{aII}^*$
Re-stabilization tongue

unstable wavelengths

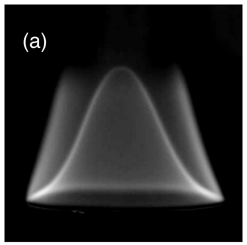



see §§ 2.5.5, 2.9.2-4 in **P.Clavin & G.Searby** (2016) Cambridge University Press

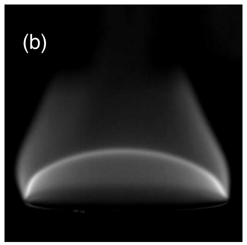
## Sensitivity of the acoustic instability to the Markstein number an the acoustic frequency



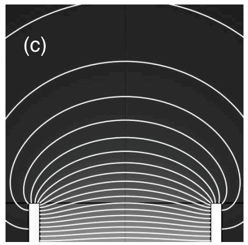

## Flattening of Bunsen flames in an acoustic field (Hahnemann Ehret 1943, Durox et al. 1997, Baillot et al. 1999)




Rich Bunsen methane flame




+ intense axial acoustic field
140 Hz




acoustic equipotential surface

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS


## Lecture IX
## Turbulent flames

# Lecture 9 : Turbulent flames

## 9-1. Introduction

## 9-2. Turbulent diffusion

*Einstein-Taylor's diffusion coefficient*

*Rough model of turbulent transport*

*Well-stirred flame regime*

## 9-3. Strongly corrugated flamelet regime

*Kolmogorov's laws*

*Gibson's scale*

*Elements of fractal geometry*

*Self similarity of strongly corrugated flames*

*Co-variant laws*

## 9-4. Turbulent combustion noise

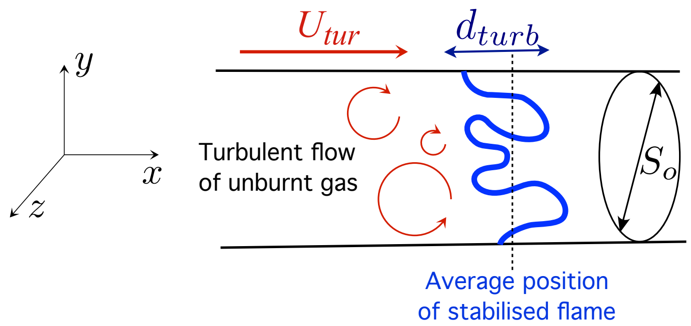
*Monopolar sound emission*

*Sound generated by a turbulent flame*

*Blow torch noise*

# 9-1. Introduction

**The problem of premixed flames in a turbulent flow is still widely open**




$U_{turb}$ ?

$d_{turb}$ ?

**Experiments are difficult. Experimental data are very scattered**

Even the simplest model has no known solution (Nonlinear stochastic equation)

*Reaction-diffusion wave in a turbulent flow* (no gas expansion)

$$\partial\theta/\partial t + \mathbf{v}(\mathbf{r},t).\nabla\theta - D_T\Delta\theta = \omega'(\theta)/\tau_{rb}. \qquad \nabla.\mathbf{v} = 0$$

prescribed turbulent flow (stochastic field)

*Same model in the wrinkled flame regime* $(l_{tur} \gg d_L, \;\; \tau_{tur} \gg \tau_L \Rightarrow \; U_n = U_L)$

eq. flame surface $\quad G(\mathbf{r},t) = G_0 \qquad \partial G/\partial t + (\mathrm{d}\mathbf{r}/\mathrm{d}t).\nabla G = 0 \qquad \mathrm{d}\mathbf{r}/\mathrm{d}t = \mathbf{v}(\mathbf{r},t) - U_n\mathbf{n} \qquad \mathbf{n} = \nabla G/|\nabla G|$

$\mathbf{v} = (u, w_y, w_z)$

$x = \alpha(y,z,t)$

$G - G_0 = x - \alpha(y,z,t)$

stochastic eikonal eq.

$$\boxed{\partial G/\partial t + \mathbf{v}(\mathbf{r},t).\nabla G = U_n|\nabla G|}$$

$$\nabla.\mathbf{v} = 0 \qquad \frac{\partial}{\partial t} < G >= U_n < |\nabla G| >$$

$$\partial\alpha/\partial t - u(\mathbf{r}_f,t) + \mathbf{w}(\mathbf{r}_f,t).\nabla_{yz}\alpha = U_{tur} - U_n\sqrt{1+|\nabla_{yz}\alpha|^2}$$

$$\langle S\rangle = \iint \mathrm{d}x\mathrm{d}y \left\langle \sqrt{1+|\nabla_{yz}\alpha|^2} \right\rangle \qquad U_{tur}/U_L = \left\langle \sqrt{1+|\nabla_{yz}\alpha|^2} \right\rangle \qquad \boxed{U_{tur}S_o = U_L\,\langle S\rangle}$$

Unfortunately the condition of existence of $< S >$ is not known ! $\quad \lim_{\Delta t\to\infty} \int_{t-\Delta t/2}^{t+\Delta t/2} S(t')\mathrm{d}t' = ?$

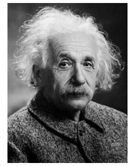
$|\mathbf{v}| \ll U_L: \quad U_{tur}/U_L \approx 1 + (|\mathbf{v}|/U_L)^2 \qquad |\mathbf{v}| \gg U_L: \quad U_{tur} \approx |\mathbf{v}|$ $\qquad$ Bending effect

P. Clavin & F.A. Williams (1979) J. Fluid Mech. 90 (3) pp. 589-604 $\qquad$ (Damköler 1940) $\qquad$ modification to the laminar flame struture

# 9-2. Turbulent diffusion

## Taylor's diffusion coefficient (analogy with Einstein random walk for molecular diffusion)

G.I Taylor 1922



Einstein 1905

1-D for simplicity: $\mathrm{d}x/\mathrm{d}t = v(t)$, $x(t) = \int_0^t v(t')\mathrm{d}t'$ $\quad v(t) = \mathrm{v}(x(t), t)$

stochastic Lagrangian velocity

ensemble average

$$\langle x^2(t)\rangle = \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \langle v(t')v(t'')\rangle \qquad \langle x^2(t)\rangle = 2\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}\tau \langle v(t')v(t'-\tau)\rangle$$

turbulence: homogeneous in time $\langle v(t)v(t-\tau)\rangle = \langle v^2\rangle g(\tau)$ $\quad g(0)=1$, $\lim_{\tau\to\infty} g = 0$

correlation function

$$\boxed{\tau_I \equiv \int_0^\infty g(\tau)\mathrm{d}\tau}$$

integral time scale

integration by parts $\quad \langle x^2(t)\rangle = 2\langle v^2\rangle \int_0^t (t-\tau)g(\tau)\mathrm{d}\tau$ $\quad$ where $\int_0^\infty \tau g(\tau)\mathrm{d}\tau = O(\tau_I^2)$ $\quad t \gg \tau_I$ : $g = 0$

first moment

$$\lim_{t\to\infty} \int_0^t (t-\tau)g(\tau)\mathrm{d}\tau = t\int_0^\infty g(\tau)\mathrm{d}\tau - \cancel{\int_0^\infty \tau g(\tau)\mathrm{d}\tau}$$

$t \gg \tau_I$, 1-D : $\langle x^2(t)\rangle = 2D_{tur}t$, $\quad$ 3-D : $\langle x^2(t)\rangle = 6D_{tur}t$, $\quad$ where

turbulent diffusion coefficient

$$\boxed{\begin{aligned} D_{tur} &\equiv \langle v^2\rangle \tau_I \\ \langle v^2\rangle &= (\text{turbulence intensity})^2 \end{aligned}}$$

## Rough model for the turbulent transport (analogy with molecular diffusion)

$$\langle \mathbf{v}\theta\rangle \approx -D_{tur}\nabla\langle\theta\rangle, \qquad \langle \nabla.(\mathbf{v}\theta)\rangle \approx -D_{tur}\Delta\langle\theta\rangle$$

limited to scalar mixing with small displacement / size (blobs, sheets ..) $l_I \ll L$ $\quad \left(v_I \approx \langle v^2\rangle^{1/2};\ l_I \equiv v_I\tau_I\right)$

turbulence intensity $\quad$ integral length scale

$$\boxed{D_{tur} = l_I v_I}$$

## Well-stirred flame regime of Damköhler (1940) $\quad l_I \ll d_L$ and $D_{tur} \gg D_T$

little practical importance

$$U_{tur} \approx \sqrt{D_{tur}/\tau_b} \qquad \frac{U_{tur}}{U_L} \approx \sqrt{\left(\frac{l_I}{d_L}\right)\left(\frac{v_I}{U_L}\right)} \gg 1, \qquad d_{tur} \approx D_{tur}/U_{tur} \gg d_L$$

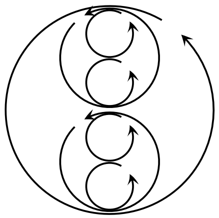
# 9-3. Strongly corrugated flamelet regime

Kolmogorov 1941

## Kolmogorov's laws

homogeneous, isotropic and fully developed turbulence

**Richardson cascade**



schematic representation

$l_i = l_I/2^i$

Decomposition in a continuos set of vortices

$l_i, \quad \tau_i, \quad v_i \equiv l_i/\tau_i$ turn-over velocity

$\mathrm{Re}_i \equiv l_i v_i/\nu$ local Reynolds nb

$\nu \equiv \mu/\rho$ viscous diffusion coeff

Kolmogorov scale $\quad l_K,\ \tau_K,\ v_K \quad \mathrm{Re}_K = 1 \quad l_i > l_K \quad v_i > v_K \quad \forall i$

Integral scale $\quad l_I,\ \tau_I,\ v_I \quad \mathrm{Re}_I \gg 1 \quad l_I > l_i \quad v_I > v_i \quad \forall i$

**Scaling laws (dimensional analysis)** $\ l_K \ll l_i \ll l_I$

energy transfer in NS eqs : $\rho(\mathbf{v}.\nabla)v^2/2 \qquad {v_i}^3/l_i \equiv \epsilon \approx \text{cst} \ \Rightarrow \ \boxed{v_i \approx \epsilon^{1/3} l_i^{1/3}, \quad v_i^2 \approx \epsilon^{2/3} l_i^{2/3}, \quad \tau_i \approx \epsilon^{-1/3} l_i^{2/3}}$

dissipation rate of energy : $\nu \mathbf{v}.\Delta\mathbf{v} \ \Rightarrow \ \epsilon = \nu\, v_K^2/l_K^2 \qquad \epsilon = v_I^3/l_I$

$\mathrm{Re}_K \equiv v_K l_K/\nu = 1 \ \Rightarrow \ \boxed{l_I/l_K \approx \mathrm{Re}_I^{3/4}, \quad v_I/v_K \approx \mathrm{Re}_I^{1/4}, \quad \tau_I/\tau_K \approx \mathrm{Re}_I^{1/2}} \qquad \mathrm{Re}_I \gg 1$

energy spectrum : $\langle v^2 \rangle /2 = \int_0^\infty \mathrm{d}k\, E(k) \qquad E(k) \approx \epsilon^{2/3} k^{-5/3} \qquad K41$ scaling law

**definition of strongly corrugated flames**

$\boxed{v_K \ll U_L \ll v_I} \Rightarrow d_L \ll l_\mathrm{K}, \quad \tau_L \ll \tau_\mathrm{K}$ no modification to the laminar flame structure

$v_K \approx \nu/l_K\,, \quad U_L \approx D_T/d_L,, \quad D_T \approx \nu$

## Gibson scale $l_G$ (Peters 1986)

definition of the Gibson scale: smallest size of the wrinkles on the flame front

turn-over time = transit time across the vortex $\quad \tau_i \approx l_i/U_L \ \Rightarrow \ v_i \approx U_L$

$\boxed{l_G \equiv U_L^3/\epsilon \ \Rightarrow \ l_K \ll l_G \ll l_I}$

$l_i < l_G \Rightarrow u_i < U_L \Rightarrow l_i/U_L < \tau_i = l_i/u_i$

transit time $<$ turn-over time

many scales of wrinkles $l_G \ll l_I \Rightarrow$ fractal geometry of the flame front

**N. Peters** (2000) *Turbulent Combustion*, Cambridge Monographs on Mechanics

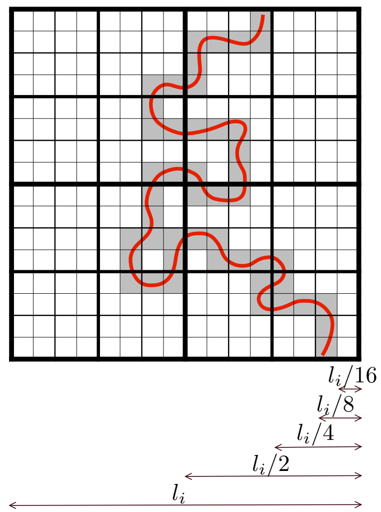
# Elements of fractal geometry

Weaker resolution $\quad l_j, \quad l_G < l_j < l_i$ $\qquad S_{i,j} \approx N_{i,j}\, l_j^2$

nb of cubes of size $l_j$ that intersect the surface within the volume $l_i^3$

Total surface area in a cube of size $l_i$, $\quad l_G < l_i < l_I$ $\qquad S_i \approx N_{i,G}\, l_G^2.$

nb of cubes of size $l_G$ that intersect the surface within the volume $l_i^3$




Fractal dimension $\mathrm{D}_f > 2:\ N_{i,j} \approx (l_i/l_j)^{D_f}$ , $\quad \boxed{S_{i,j}/l_i^2 \approx (l_i/l_j)^{D_f - 2}}$

Regular surface: $D_f = 2 \Rightarrow$ total area $S_i$ in a box of size $l_i$ $\qquad S_i/l_i^2 = \lim\limits_{l_j \to 0} S_{i,j}/l_i^2 =$ finite cst.

For a flame of thickness $d_L$ its area is well defined for wrinkles whose scale is larger than $d_L$, $\ l_j > d_L$

The fractal dimension $D_f > 2$ can concern only scales greater than the smallest wrinkles

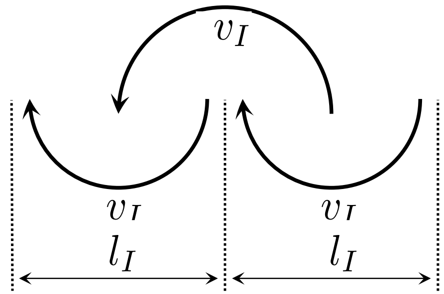
Fractal dimension of a turbulent flame can be meaningful only for $\quad d_L < l_G < l_j < l_i < l_I$

# Self similarity of strongly corrugated flames

$\boxed{v_K \ll U_L \ll v_I \quad d_L \ll l_K \ll l_G \ll l_I}$

Assumption: the Kolmogorov scaling law is not modified by the gas expansion ok for $l_i \gg l_G$

## *Contamination time vs combustion time*




Kolmogorov scaling law : $\boxed{\tau_i \approx \epsilon^{-1/3} l_i^{2/3} \searrow l_i \searrow \qquad v_i \approx \epsilon^{1/3} l_i^{1/3} \nearrow \ l_i \nearrow}$

Fastest contamination: integral scale $v_I \gg v_i$. $U_{tur} = v_I$?

ok if the combustion time of the vortex is not longer than the turnover time

## *Self similar law*

An effective front of thickness $l_i$ is defined at each scale

A flame velocity $U_i$ can be defined at each scale if $U_i = (S_{i,j}/l_i^2)U_j \qquad U_i/U_j = \langle S_{i,j} \rangle / l_i^2$

At the Gibson scale the combustion time of the vortex = turnover time $\quad U_L = v_G \quad \Rightarrow \quad l_G/U_L = l_G/v_G$

Self similarity: same law at all scales $\Rightarrow$ $\boxed{\text{combustion time of the vortex = the turnover time } \forall\, l_i}$

$$l_i/U_i = \tau_i \quad \Rightarrow \quad U_i = v_i$$

Kolmogorov cascade $\Rightarrow$ small vortices burn faster than larger ones

$$\boxed{U_{tur} = v_I, \quad l_{tur} = l_I}$$

Fractal dimension of the flame surface: Kolmogorov scaling $v_i/v_j = (l_i/l_j)^{1/3}$

$$U_i/U_j = \langle S_{i,j} \rangle / l_i^2 \underset{U_i = v_i}{\Rightarrow} v_i/v_j = \langle S_{i,j} \rangle / l_i^2 \Rightarrow \langle S_{i,j} \rangle / l_i^2 = (l_i/l_j)^{1/3} \Rightarrow \boxed{D_f = 7/3}$$

$$\langle S_{i,j} \rangle / l_i^2 = (l_i/l_j)^{D_f - 2}$$

def fractal dimension The result is the same for all mixtures...??

# Co-variant laws

A. Pocheau (1994) *Phys. Rev. E*, **49**, 1109-1122

More general law independent of the turbulent scaling and satisfying additivity

Turbulent energy contained in the range $[l_i, l_j]$ : $\quad v_{i,j}^2 \equiv \sum_{k=i}^{j-1} v_k^2 \qquad v_k^2$ : energy in $[k, k+1]$

Co-variant law = same for each couple of length scales $l_i,\ l_j \qquad l_i > l_j$

The only co-variant law for the flame velocity $U_i$ at scale $l_i$ satisfying additivity is $\quad U_i^2 = U_j^2 + c\, v_{i,j}^2$

Pocheau 1994

Co-variance ? $\qquad l_i > l_k > l_j, \quad v_{i,j}^2 = v_{i,k}^2 + v_{k,j}^2 \qquad U_i^2 = U_j^2 + c\, v_{i,k}^2 + c\, v_{k,j}^2 = U_k^2 + c\, v_{i,k}^2$

co-variance ok $\quad U_i^2 = U_k^2 + c\, v_{i,k}^2$

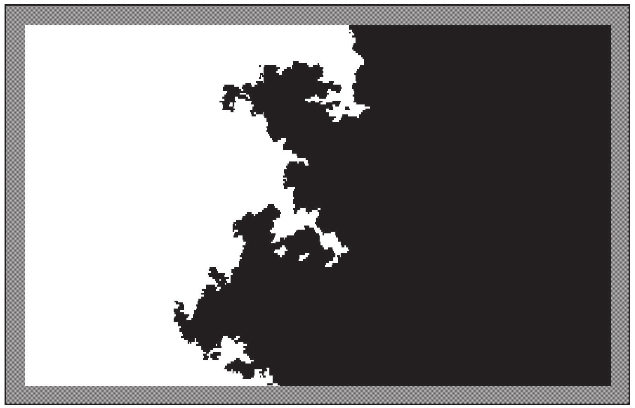
$$\boxed{U_{tur}^2 = U_L^2 + c\, v^2} \qquad v^2 \equiv \sum_{\forall n} v_n^2 \quad \text{turbulence intensity}$$

Not limited to a strong turbulence

The case $c = 1$ covers the known results at low and large turbulence intensity

Reasonably good agreement with experiments

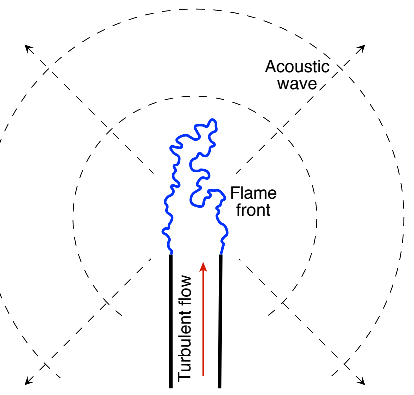
$$v/U_L = O(1), \quad l_I/l_K \approx 180$$



A. Pocheau & D. Quieros-Condé (1996) *Phys. Rev. Lett*, **76** (18) 3352-3355

# 9-4. Turbulent combustion noise

wavelength $a/\omega \gg L$ size of the flame




## Monopolar sound emission

Deformable (small) body with fluctuating volume $V(t)$

$$\mathbf{u} = \nabla\phi(\mathbf{r},t) \quad \text{acoustic potential } \phi(\mathbf{r},t) = -\frac{\dot{V}(t-a\,r)}{4\pi r} \qquad r \equiv |\mathbf{r}|, \quad \dot{V}(t) \equiv \mathrm{d}V/\mathrm{d}t$$

$$r \gg L: \quad v = (4\pi a r)^{-1}\ddot{V}(t-r/a), \quad \ddot{V}(t) \equiv \mathrm{d}^2V(t)/\mathrm{d}t^2$$

Radiated flux of energy (intensity of sound) $\quad = \quad \rho a\left\langle v^2\right\rangle$

Total acoustic energy radiated/unit time $I \quad = (\rho/4\pi a)\left\langle\left(\mathrm{d}^2V/\mathrm{d}t^2\right)^2\right\rangle$

## Sound generated by a turbulent flame $\quad \mathrm{d}V/\mathrm{d}t = \dot{M}_b/\rho_b$

mass flow rate of burned gas in the lab frame

mass flow rate of fresh gas in the lab frame

$$\dot{M}_b = \rho_b \iint_S (\mathcal{D}_f + U_b)\mathrm{d}^2\sigma \qquad \dot{M}_u = \rho_u \iint_S (\mathcal{D}_f + U_L)\mathrm{d}^2\sigma \qquad \rho_u U_L = \rho_b U_b$$

normal flame velocity in the lab frame

elimination of $\mathcal{D}_f$ $\qquad \mathrm{d}V/\mathrm{d}t = \dot{M}_b/\rho_b = \dot{M}_u/\rho_u + (U_b - U_L)S \qquad \boxed{\mathrm{d}^2V/\mathrm{d}^2t = (U_b - U_L)\mathrm{d}S/\mathrm{d}t}$

constant

$$I = (\rho/4\pi a)(U_b - U_L)^2\left\langle(\mathrm{d}S/\mathrm{d}t)^2\right\rangle \qquad \mathrm{d}\tilde{I}(\omega)/\mathrm{d}\omega = \frac{\rho}{4\pi a}(U_b - U_L)^2\int_0^\infty \mathrm{d}t\,\mathrm{e}^{\mathrm{i}\omega t}\left\langle\dot{S}(t)\dot{S}(0)\right\rangle$$

intensity of sound — power spectrum of sound

Strahle 1985

$\dot{S}(t) \equiv \mathrm{d}S/\mathrm{d}t$

total intensity of sound

$$I = (\rho/4\pi a)(U_b - U_L)^2 \left\langle (\mathrm{d}S/\mathrm{d}t)^2 \right\rangle$$

power spectrum of sound

$$\mathrm{d}\tilde{I}(\omega)/\mathrm{d}\omega = \frac{\rho}{4\pi a}(U_b - U_L)^2 \int_0^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \left\langle \dot{S}(t)\dot{S}(0) \right\rangle$$

Strongly corrugated regime with the Kolmogorov scaling

$$D_f = 7/3 \quad \Rightarrow \quad I \approx \frac{1}{4\pi}\left(\frac{T_b}{T_u} - 1\right)^2 (\rho \Delta V)\frac{v_I^4}{a l_I}$$

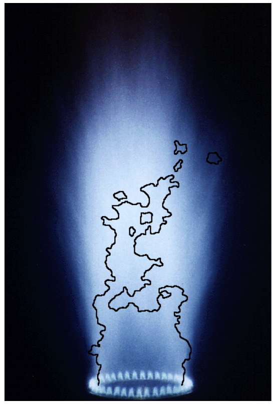
total volume of the flame brush

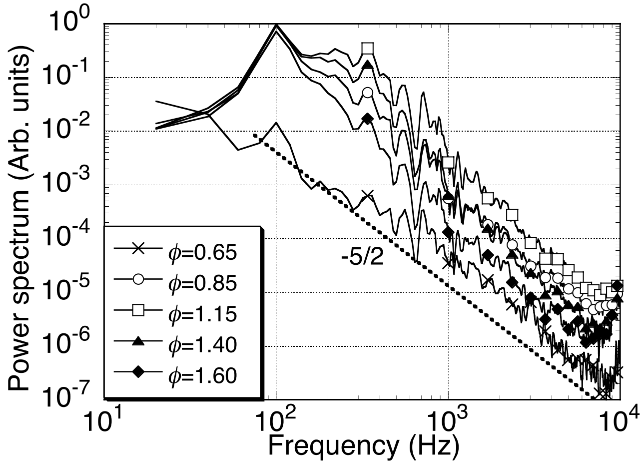
$$\mathrm{d}\tilde{I}(\omega) \propto \omega^{-5/2}\mathrm{d}\omega$$

Clavin, Siggia (1991) *Combust. Sci. Technol.* **78**, 147-155

in agreement with experiments on very large burners

(Abugov Obrezkov 1978)





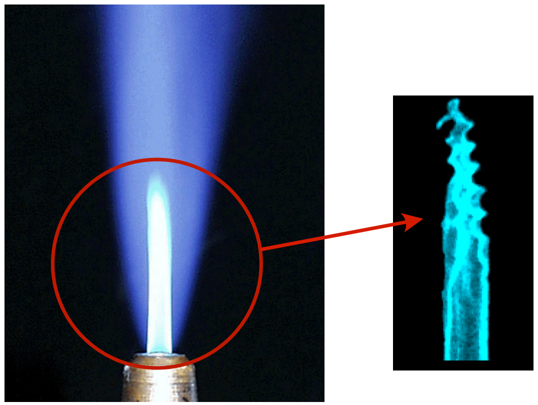

## Blowtorch noise

Combustion noise is two orders of magnitude higher

the noise is not resulting from the direct interaction of upstream turbulence on the flame front

amplification by the intrinsic flame instability is essential

Searby et al. 2001 *Phys. Fluids.* ***13***, 3270-3275



*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

## Lecture X
## Supersonic waves (shocks and detonations)

# Lecture 10 : Supersonic waves

## 10-1. Background

*Model of hyperbolic equations for the formation of discontinuity*

*Riemann invariants*

*Rankine-Hugoniot conditions for shock waves*

## 10-2. Inner structure of a weak shock wave

*Formulation*

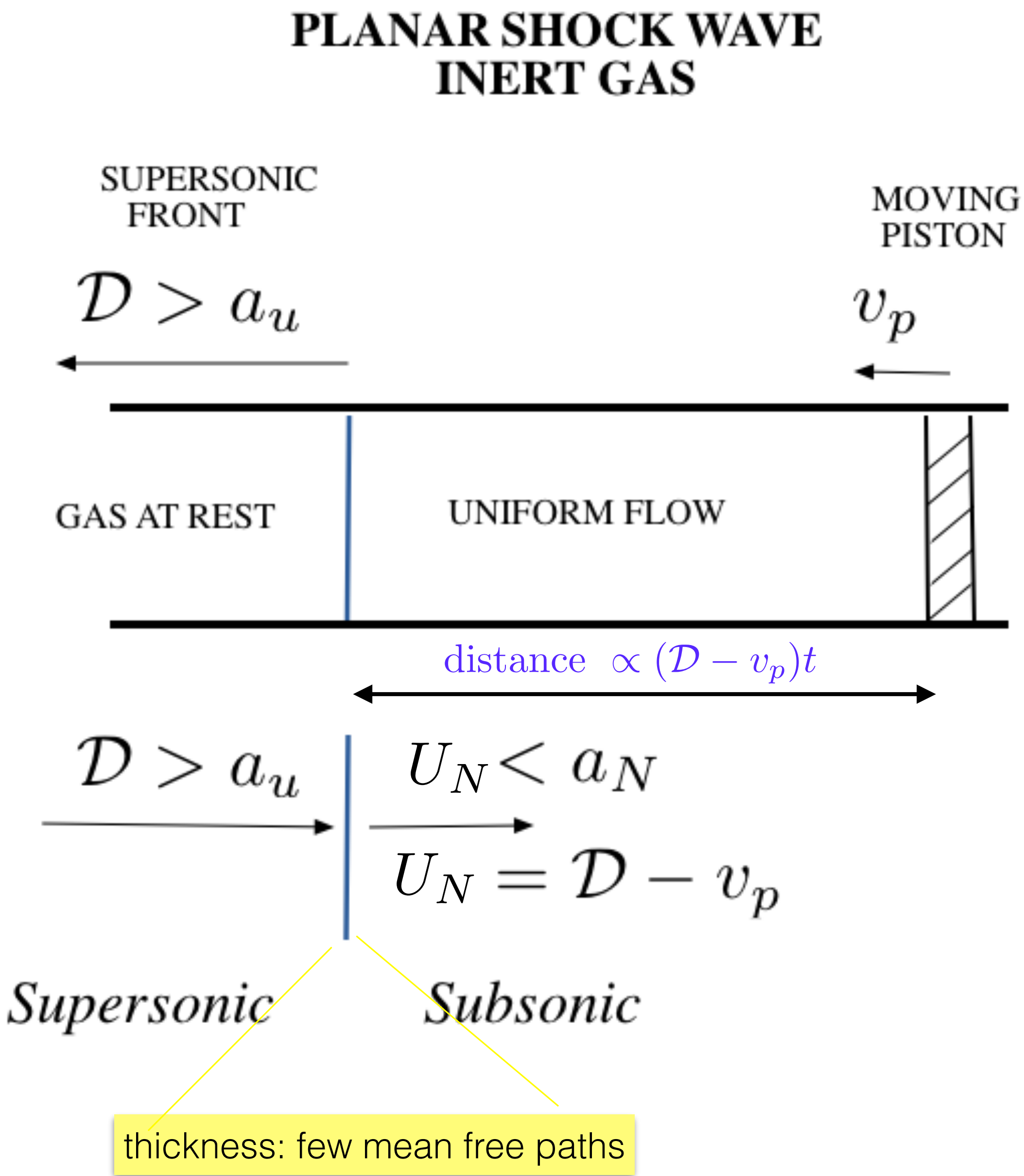
*Dimensional analysis*

*Analysis*

## 10-3. ZND structure of detonations

## 10-4. Selection mechanism of the CJ wave

# 10-1. Background




Poisson 1808, Stokes 1848, **Riemann** 1860, Rankine 1869, **Hugoniot** 1889, Rayleigh 1910

# SHOCK WAVE AS A SINGULARITY OF THE EULER EQUATIONS

*Riemann (1860)*

shock wave $\approx$ discontinuity in the solution of the Euler equations

# Model of hyperbolic equations for the formation of discontinuities

**Nonlinear** first order p**artial - differential** equation

$u(x,t)$? $\boxed{\partial u/\partial t + a(u)\partial u/\partial x = 0}$ $a(u)$ given function

initial condition

$t = 0:\quad u = u_o(x)$ $u = u(x,t)$ ?

---

Simple case: linear equation

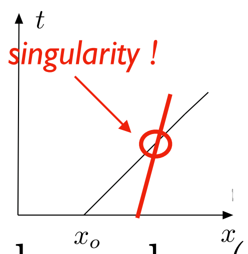
$a = a_o = \text{cst.}:$ $\boxed{u = u_o(x - a_o t)}$

propagation at constant velocity without deformation

---

Nonlinear equation $a(u)$ $\mathrm{d}a/\mathrm{d}u \neq 0$

***Method of characteristics***




Riemann 1860

The solution is conserved along any trajectory $\mathrm{d}x/\mathrm{d}t = a(u)$ in the phase plan $(x,t)$.

$u = u(x(t),t)$ $\mathrm{d}u/\mathrm{d}t = 0 \quad \Rightarrow \quad \mathrm{d}x/\mathrm{d}t =$ constant

$u(x,t)$ is constant along the straight lines $\boxed{x = a_o t + x_o}$: $u = u_o$ $u_o \equiv u(a_o)$

$du_0/dx_0 \neq 0 \quad \Rightarrow \quad da_0/dx_0 \neq 0$ Intersection of characteristics: finite time singularity

Nonlinear equation $\boxed{\partial u/\partial t + a(u)\partial u/\partial x = 0}$

$t = 0: \quad u = u_o(x)$ $\qquad u = u(x,t)$ ?

***Method of characteristics***

Riemann 1860

trajectories $x(t)$ : $\quad \mathrm{d}x/\mathrm{d}t = a(u) \quad u = u(x(t),t) \quad \mathrm{d}u/\mathrm{d}t = 0$

$$u(x,t) = \text{cst.} \Rightarrow a(x,t) = \text{cst.}$$

$u(x,t)$ is constant along the straight lines; $\quad t = 0: \; x = a_o t + x_o, \; u = u_o$

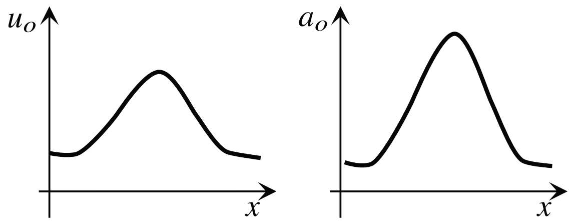
$$\boxed{x = a_o t + x_o} \qquad u_o \equiv u(a_o)$$

---

Speed increases with increasing $u$, $\boxed{\mathrm{d}a/\mathrm{d}u > 0}$

$$u_o(x) \equiv u(x, t=0)$$

Initial state

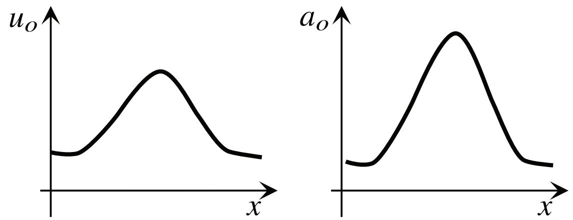



$\mathrm{d}a/\mathrm{d}u > 0$ : larger values of $u$ run faster

$\Rightarrow$ $\boxed{\text{formation of singularities after a finite time}}$

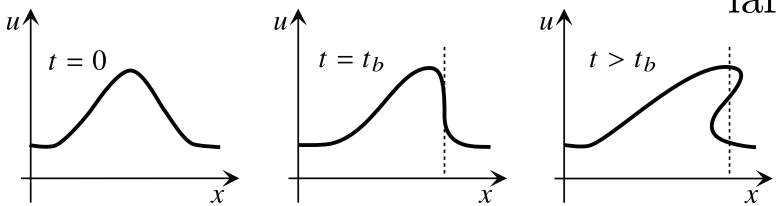
$$\boxed{\partial u/\partial t + a(u)\partial u/\partial x = 0}$$

Speed increases with increasing $u$, $\boxed{\mathrm{d}a/\mathrm{d}u > 0}$



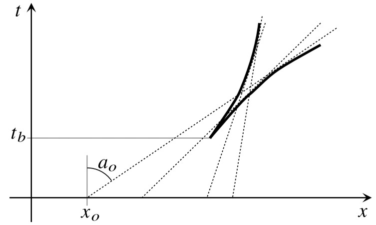

initial condition

$\Rightarrow$ $\boxed{\text{formation of singularities after a finite time}}$




$t > t_b$ : multivalued solution. Wave breaking

larger values run faster




$t > t_b$ : characteristics intersect

$u(x,t)$ constant on straight trajectories in the phase plane $(t,x)$

$$x(x_0,t) = x_0 + a_0(x_0)t: \quad u(x,t) = u_0(x_0)$$

$$\partial x/\partial x_o = 1 + t(\mathrm{d}a_o/\mathrm{d}x_o) \qquad \partial x_o/\partial x = [1 + t(\mathrm{d}a_o/\mathrm{d}x_o)]^{-1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial x_o}{\partial x}\frac{\mathrm{d}u_o}{\mathrm{d}x_o} \quad \frac{\partial u}{\partial x} = \frac{\mathrm{d}u_o/\mathrm{d}x_o}{[1 + t(\mathrm{d}a_o/\mathrm{d}x_o)]} \quad \text{diverges at time} \quad t = \frac{1}{-\mathrm{d}a_o/\mathrm{d}x_o} \quad \text{where} \quad da_o/\mathrm{d}x_o < 0$$

$t_b \equiv$ **time of wave breaking** (shortest time for the divergence of $\partial u/\partial x$)

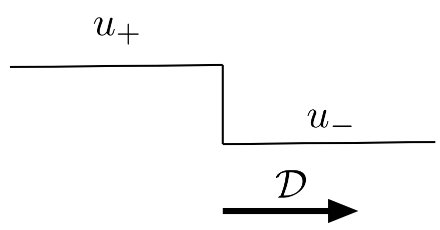
$$\boxed{t_b = \frac{1}{\max|\mathrm{d}a_0/\mathrm{d}x_0|}}$$

$\partial u/\partial t + a(u)\partial u/\partial x = 0$ conservative form $\partial u/\partial t + \partial j/\partial x = 0$

$j(u)$ $\mathrm{d}j/\mathrm{d}u = a(u)$ $j(u) \equiv \int_{u_-}^{u} a(u')\mathrm{d}u'$

# **Discontinuous** solution

Are step functions $u_+ \neq u_-$ propagating at constant velocity $\mathcal{D}$ solutions ?




$$\boxed{u(\xi) \qquad \xi = x - \mathcal{D}t}$$

$$\partial/\partial t = -\mathcal{D}\mathrm{d}/\mathrm{d}\xi \qquad \partial/\partial x = \mathrm{d}/\mathrm{d}\xi$$

$\mathcal{D}?$

$$-\mathcal{D}\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{\mathrm{d}j}{\mathrm{d}\xi} = 0 \qquad j - \mathcal{D}u \quad \text{is a conserved scalar}$$

$$j(u_+) - \mathcal{D}u_+ = j(u_-) - \mathcal{D}u_- \qquad \mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-}$$

---

Infinite numbers of solutions !! *Ill posed problem*

$$f(u) \times \left(-\mathcal{D}\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{\mathrm{d}j}{\mathrm{d}\xi}\right) = 0 \quad \forall \text{ function } f(u)$$

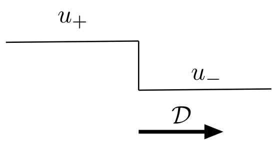
Definition of $F(u)$ and $G(u)$ : $\frac{\mathrm{d}F}{\mathrm{d}u} \equiv f(u) \qquad \frac{\mathrm{d}G}{\mathrm{d}u} \equiv f(u)a(u)$

$$\frac{\mathrm{d}}{\mathrm{d}u}[-\mathcal{D}F(u) + G(u)] = 0 \quad \Rightarrow \quad \mathcal{D} = \frac{G(u_+) - G(u_-)}{F(u_+) - F(u_-)}$$

# *Discontinuous solutions*

$$\partial u/\partial t + a(u)\partial u/\partial x = 0$$

Are step functions $u_+ \neq u_-$ propagating at constant velocity $\mathcal{D}$ solutions ?

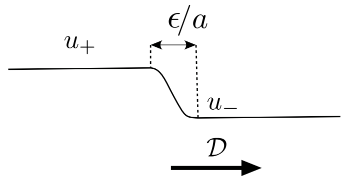



$$\boxed{u(\xi) \qquad \xi = x - \mathcal{D}t}$$

Infinite numbers of solutions !! ***Ill posed problem***

---

*adding a dissipative term makes the problem well posed*




$$\boxed{u(\xi) \qquad \xi = x - \mathcal{D}t} \qquad \partial u/\partial t + a(u)\partial u/\partial x = \boxed{\epsilon\partial^2 u/\partial x^2,} \quad \epsilon > 0$$

$$\partial/\partial t = -\mathcal{D}\mathrm{d}/\mathrm{d}\xi \qquad \partial/\partial x = \mathrm{d}/\mathrm{d}\xi \qquad \frac{\partial u}{\partial t} \to -\mathcal{D}\frac{\mathrm{d}u}{\mathrm{d}\xi}$$

dimension $\epsilon = \text{length}^2/\text{time}$

$$j(u) \equiv \int_{u_-}^{u} a(u')\mathrm{d}u' \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\left[-u\mathcal{D} + j(u) - \epsilon\frac{\mathrm{d}u}{\mathrm{d}\xi}\right] = 0 \qquad \epsilon\frac{\mathrm{d}u}{\mathrm{d}\xi} = j(u) - u\mathcal{D} + \mathrm{cst}$$

$$\frac{\xi}{\epsilon} = \int_0^u \frac{\mathrm{d}u}{j(u) - u\mathcal{D} + \mathrm{cst}}$$

$\xi = \pm\infty :\ \mathrm{d}u/\mathrm{d}\xi = 0 \ \Rightarrow$ 2 expressions of the cst that should be equal $\Rightarrow$ a single value of $\mathcal{D}$

$$j(u_+) - u_+\mathcal{D} + \mathrm{cst} = 0$$
$$j(u_-) - u_-\mathcal{D} + \mathrm{cst} = 0$$

$$\boxed{\mathcal{D} = \frac{j(u_+) - j(u_-)}{u_+ - u_-}}$$

independent of $\epsilon$ !

$u(\xi)$ continuous function $\qquad \lim_{\epsilon \to 0} u =$ step function

$a(u) = u :$ Burgers equation. Analytical solution to the initial value problem in G..B. Whitham (1974) John Wiley & Sons, chapter 4.

# Riemann invariants (1860)

Euler equation + constant entropy + ideal gas: $a^2 = \dfrac{\mathrm{d}p}{\mathrm{d}\rho} = \gamma \dfrac{p}{\rho}$ $\quad \dfrac{p}{\rho^\gamma} = \text{cst} \quad 2\dfrac{\mathrm{d}a}{a} = \dfrac{\mathrm{d}p}{p} - \dfrac{\mathrm{d}\rho}{\rho} \Rightarrow \boxed{2\dfrac{\mathrm{d}a}{a} = (\gamma - 1)\dfrac{\mathrm{d}\rho}{\rho}}$

$$\lambda \times \underbrace{\left[\frac{\partial}{\partial t}\rho + u\frac{\partial}{\partial x}\rho + \rho\frac{\partial}{\partial x}u = 0\right]}_{\text{mass}} + \underbrace{\overbrace{\frac{a^2}{\rho}\frac{\partial}{\partial x}\rho}^{(1/\rho)\partial p/\partial x} + \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u = 0}_{\text{momentum eq. }/\rho}$$

$$\lambda\frac{\partial}{\partial t}\rho + \left(\lambda u + \frac{a^2}{\rho}\right)\frac{\partial}{\partial x}\rho + \frac{\partial}{\partial t}u + (\lambda\rho + u)\frac{\partial}{\partial x}u = 0$$

Riemann 1860

choose $\lambda$ such that $\quad \lambda u + a^2/\rho = \lambda(\lambda\rho + u), \quad \Rightarrow \quad \lambda = \pm a/\rho \qquad \lambda\rho = \pm a$

$$\lambda\left[\frac{\partial}{\partial t} + (\lambda\rho + u)\frac{\partial}{\partial x}\right]\rho + \left[\frac{\partial}{\partial t} + (\lambda\rho + u)\frac{\partial}{\partial x}\right]u = 0$$

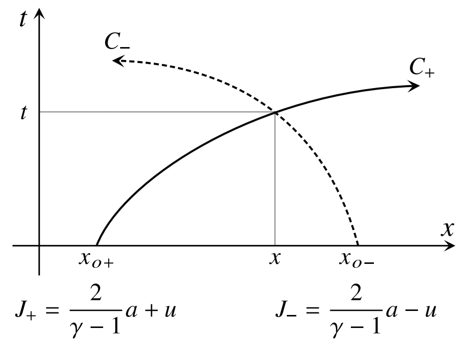
$\lambda\rho + u = u \pm a$

2 characteristics : $\quad C_+ : \ \dfrac{\mathrm{d}x}{\mathrm{d}t} = u + a, \qquad C_- : \ \dfrac{\mathrm{d}x}{\mathrm{d}t} = u - a$

invariants : $\quad C_+ : \ \dfrac{a}{\rho}\mathrm{d}\rho + \mathrm{d}u = 0 \qquad C_- : \ \dfrac{a}{\rho}\mathrm{d}\rho - \mathrm{d}u = 0$

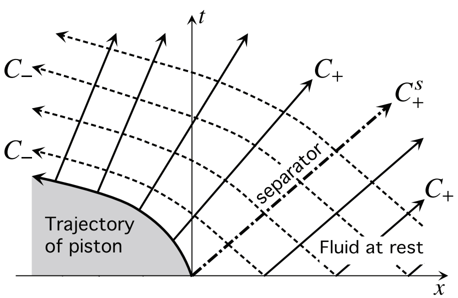
$\lambda \mathrm{d}\rho + \mathrm{d}u = 0 \qquad \pm\dfrac{a}{\rho}\mathrm{d}\rho + \mathrm{d}u = 0$

$$\boxed{J_+ \equiv \frac{2}{\gamma - 1}a + u = \ \text{cst sur } C_+ \qquad J_- \equiv \frac{2}{\gamma - 1}a - u = \ \text{cst sur } C_-}$$

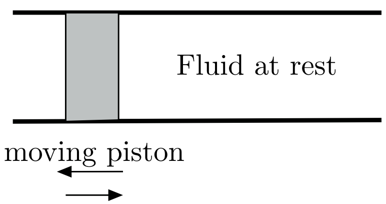


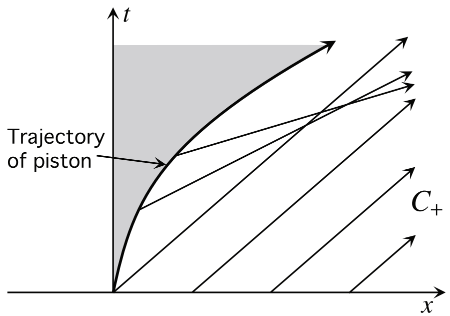

***Simple waves***

$C_+$ are straight lines




Rarefaction wave




Compression wave

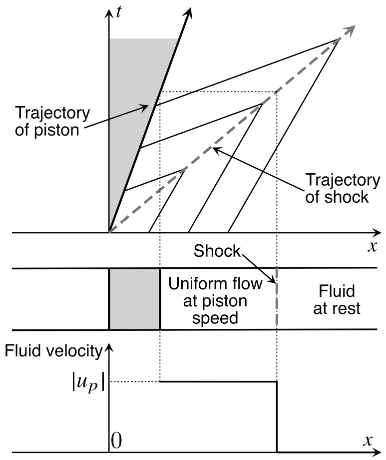
formation of a singularity: shock wave

# *Centred waves*

$$t < 0: \quad \text{piston velocity } = 0$$
$$t > 0: \quad \text{piston velocity } = \text{ cst } \neq 0$$

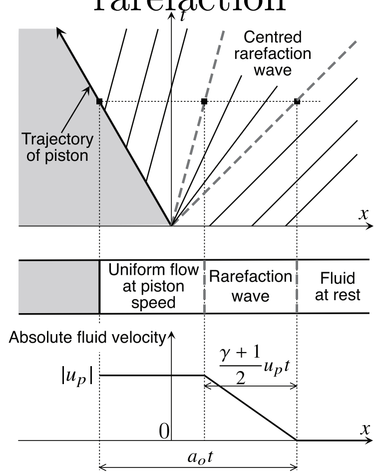
Sel-similar solutions $x/t$

compression




shock wave at constant velocity $>$ piston velocity

rarefaction




no discontinuity of u

discontinuity of du/dx propagating at the local sound speed

fully unsteady process: thickness $\nearrow$ $u_p\, t$

Rankine-Hugoniot conditions

(jumps across a planar shock wave)

# Planar shock waves

Rankine 1870

Hugoniot 1880

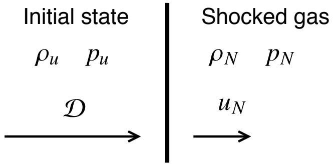
## Rankine-Hugoniot conditions for shock waves

(1870 – 1880)

Eqs for the conservation of mass, momentum and energy

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \qquad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x}\left(p + \rho u^2 - \mu\frac{\partial u}{\partial x}\right) \qquad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u(h + u^2/2) - \lambda\frac{\partial T}{\partial x} - \mu u\frac{\partial u}{\partial x}\right]$$

Inner structure in steady state: working in the reference frame of the wave; **equality of the fluxes on both sides**




written in the moving frame of the shock at velocity $\mathcal{D}$

steady problem

$$\boxed{m \equiv \rho_u \mathcal{D}} = \rho_N u_N \qquad p_u + \frac{m^2}{\rho_u} = p_N + \frac{m^2}{\rho_N} \qquad h_N - h_u + (u_N^2 - \mathcal{D}^2)/2 = 0$$

$$p_u - p_N = m^2\left[\frac{1}{\rho_N} - \frac{1}{\rho_u}\right] \qquad h_u - h_N = \frac{m^2}{2}\left[\frac{1}{\rho_N^2} - \frac{1}{\rho_u^2}\right]$$

$$h(\rho_u, p_u) - h(\rho_N, p_N) + \frac{1}{2}\left(\frac{1}{\rho_u} + \frac{1}{\rho_N}\right)(p_N - p_u) = 0$$

Hugoniot curve $(p - 1/\rho)$

$$h(p,\rho) - h(p_u,\rho_u) - \frac{1}{2}\left(\frac{1}{\rho_u} + \frac{1}{\rho}\right)(p - p_u) = 0$$

Michelson-Rayleigh line

$$p - p_u = -m^2\left(\frac{1}{\rho} - \frac{1}{\rho_u}\right)$$

## *Ideal (polytropic) gas* $\gamma \equiv c_p/c_v$

$$p = (c_p - c_v)\rho T, \qquad h = c_p T = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

$$h(p,\rho) - h(p_u,\rho_u) - \frac{1}{2}\left(\frac{1}{\rho_u} + \frac{1}{\rho}\right)(p - p_u) = 0$$

$$p - p_u = -m^2\left(\frac{1}{\rho} - \frac{1}{\rho_u}\right) \qquad M_u = \frac{\mathcal{D}}{a_u} > 1$$

$$\mathcal{P} \equiv \frac{(\gamma+1)}{2\gamma}\left(\frac{p}{p_u} - 1\right), \quad \mathcal{V} \equiv \frac{(\gamma+1)}{2}\left(\frac{\rho_u}{\rho} - 1\right)$$

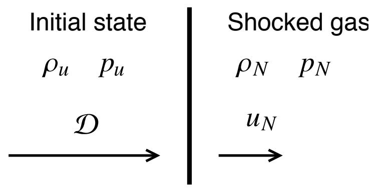
$(\mathcal{P}+1)(\mathcal{V}+1) = 1$ Hugoniot curve $\qquad \mathcal{P} = -M_u^2\mathcal{V}$ Michelson-Rayleigh line

quadratic equation for $\mathcal{V}$, 2 solutions: $\mathcal{V} = 0$, $\mathcal{V} = \mathcal{V}_N$

### *Shocked gas (Neumann state)* vs $M_u$

$$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma-1)M_u^2 + 2}{(\gamma+1)M_u^2} \qquad \frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma-1)}{(\gamma+1)} \qquad \frac{T_N}{T_u} = \frac{[2\gamma M_u^2 - (\gamma-1)]\,[(\gamma-1)M_u^2 + 2]}{(\gamma+1)^2 M_u^2}$$

$$M_N = \frac{u_N}{a_N} < 1, \quad M_N^2 = \frac{(\gamma-1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma-1)} \quad \Leftrightarrow \quad 2\gamma M_u^2 M_N^2 - (\gamma-1)(M_u^2 + M_N^2) - 2 = 0$$



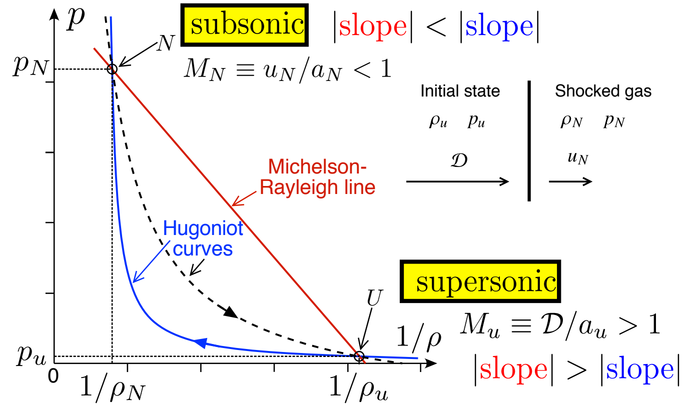

## *General comments*

*The Hugoniot curve is tangent to the isentropic*

the entropy change along the Hugoniot curve is of third order $\Rightarrow$ $M_u \equiv \mathcal{D}/a_u > 1$

$$\delta s = s - s_u \qquad \delta p = p - p_u \qquad \delta s = \frac{1}{12}\frac{1}{T_u}\left(\frac{\partial^2(1/\rho)}{\partial p^2}\right)_s (\delta p)^3$$

$$\delta s \nearrow \delta p \nearrow$$




*The Hugoniot relation is not an iso-function of state*

$h(p,\rho) - h(p_u,\rho_u) - \frac{1}{2}\left(\frac{1}{\rho_u} + \frac{1}{\rho}\right)(p - p_u) = 0$ cannot be written in the form $\mathcal{H}(1/\rho, p) = \mathcal{H}(1/\rho_u, p_u)$

$$h(p,\rho) - h(p_N,\rho_N) - \frac{1}{2}\left(\frac{1}{\rho_N} + \frac{1}{\rho}\right)(p - p_N) = 0 \neq h(p,\rho) - h(p_u,\rho_u) - \frac{1}{2}\left(\frac{1}{\rho_u} + \frac{1}{\rho}\right)(p - p_u) = 0$$

*Rarefaction shock does not exist. The entropy of the fluid increases through the shock* (Irreversibility)

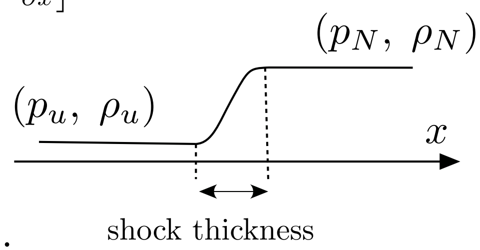
can be proved for weak shock by the entropy balance $\qquad \rho u \dfrac{\mathrm{d}s}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\lambda}{T}\dfrac{\mathrm{d}T}{\mathrm{d}x}\right) + \dot{\omega}_s \qquad \dot{\omega}_s > 0$

or by the H-theorem using the Boltzmann equation

# 10-2. Inner structure of a weak shock wave

Clavin, Searby (2014) *Cambridge University Press, pp. 219-222*

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}=0 \qquad \frac{\partial(\rho u)}{\partial t}=-\frac{\partial}{\partial x}\left(p+\rho u^2-\mu\frac{\partial u}{\partial x}\right) \qquad \frac{\partial(\rho e_{tot})}{\partial t}=-\frac{\partial}{\partial x}\left[\rho u(h+u^2/2)-\lambda\frac{\partial T}{\partial x}-\mu u\frac{\partial u}{\partial x}\right]$$




## Formulation

( reference frame attached to the shock wave )

$$\boxed{\rho u = m,} \qquad p+\rho u^2-\mu\frac{\mathrm{d}u}{\mathrm{d}x}=\text{cst.} \qquad m\left(h+\frac{u^2}{2}\right)-\lambda\frac{\mathrm{d}T}{\mathrm{d}x}-\mu u\frac{\mathrm{d}u}{\mathrm{d}x}=\text{cst.}$$

$$x\to-\infty: \quad p=p_u, \quad \rho=\rho_u, \quad u=\mathcal{D} \qquad\qquad x\to\infty: \quad \mathrm{d}p/\mathrm{d}x=0, \quad \mathrm{d}p/\mathrm{d}x=0,, \quad \mathrm{d}u/\mathrm{d}x=0$$

$$m=\rho_u\mathcal{D} \qquad \frac{p}{\rho}=(\gamma-1)c_vT \qquad h=\frac{\gamma}{\gamma-1}\frac{p}{\rho}$$

*Two coupled equations for* $p$ *and* $\upsilon\equiv 1/\rho, m$ *given* $\quad (u=m\upsilon, \quad c_vT=p\upsilon/(\gamma-1))$

$$(p-p_u)+m^2(\upsilon-\upsilon_u)=\mu m\frac{\mathrm{d}\upsilon}{\mathrm{d}x},$$

$$\frac{\gamma}{\gamma-1}(p\upsilon-p_u\upsilon_u)-\frac{1}{2}(p-p_u)(\upsilon+\upsilon_u)=\frac{\gamma}{\gamma-1}\frac{\lambda}{mc_p}\frac{\mathrm{d}(p\upsilon)}{\mathrm{d}x}+\frac{\mu m}{2}(\upsilon-\upsilon_u)\frac{\mathrm{d}\upsilon}{\mathrm{d}x}$$

$$\frac{1}{2}(u^2-u_u^2)=\frac{m^2}{2}(\upsilon+\upsilon_u)(\upsilon-\upsilon_u)$$
$$=\frac{1}{2}(\upsilon+\upsilon_u)\left[-(p-p_u)+\mu m\frac{\mathrm{d}\upsilon}{\mathrm{d}x}\right]$$

$$x\to\infty: \quad \mathrm{d}p/\mathrm{d}x=0, \quad \mathrm{d}\upsilon/\mathrm{d}x=0$$

## Dimensional analysis

speed of sound

$u/a=O(1),$

mean free path

$\mu/\rho=$ viscous diffusion coefficient $\approx \ell/a$

kinetic theory of gases

$$\rho u^2 \qquad \mu\frac{\mathrm{d}u}{\mathrm{d}x}$$

$\Rightarrow$ **thickness of shock waves $\approx$ mean free path**

macroscopic equations not valid ?

ok for weak shock !

$M_u \equiv \mathcal{D}/a_u > 1$ **Analysis for** $\boxed{\epsilon \equiv M_u - 1 \ll 1}$ (weak shock) $\boxed{\mathcal{D}/a_u = 1 + \epsilon}$

$$(p - p_u) + m^2(v - v_u) = \mu m \frac{\mathrm{d}v}{\mathrm{d}x}, \qquad m = \rho_u \mathcal{D}$$

$$\frac{\gamma}{\gamma-1}(pv - p_u v_u) - \frac{1}{2}(p - p_u)(v + v_u) = \frac{\gamma}{\gamma-1}\boxed{\frac{\lambda}{mc_p}}\frac{\mathrm{d}(pv)}{\mathrm{d}x} + \frac{\mu m}{2}\overset{\mathrm{O}(\epsilon^3)}{\cancel{(v - v_u)}}\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$x \to -\infty : p = p_u,\ v = v_u \qquad x \to \infty : \mathrm{d}p/\mathrm{d}x = 0,\ \mathrm{d}v/\mathrm{d}x = 0$$

*Non-dimensional equations*

$$\boxed{v \equiv 1/\rho \qquad \nu \equiv (v - v_u)/v_u = O(\epsilon) \qquad \pi \equiv (p - p_u)/p_u = O(\epsilon)}$$

anticipating $\begin{cases} \mathrm{d}\nu/\mathrm{d}\xi = O(\epsilon^2) \\ \mathrm{d}\pi/\mathrm{d}\xi = O(\epsilon^2) \end{cases}$

$$\lambda/mc_p = D_{Tu}/(\rho_u \mathcal{D}) \approx D_{Tu}/(\rho_u a_u) \qquad \mathrm{Pr} \equiv \mu/(\rho_u D_{Tu})$$

mean free path

$$\boxed{\xi \equiv x/\ell \qquad \ell \equiv D_{Tu}/a_u} \qquad a_u = \sqrt{\gamma p_u/\rho_u} \qquad M_u^2 = 1 + 2\epsilon + ..$$

$$\begin{cases} \dfrac{1}{\gamma}\pi + (1+2\epsilon)\nu = \mathrm{Pr}\dfrac{\mathrm{d}\nu}{\mathrm{d}\xi} + O(\epsilon^3), \\ \left(\dfrac{\gamma+1}{2\gamma}\right)\pi\nu + \dfrac{1}{\gamma}\pi + \nu = \dfrac{\mathrm{d}\pi}{\mathrm{d}\xi} + \dfrac{\mathrm{d}\nu}{\mathrm{d}\xi} + O(\epsilon^3), \end{cases}$$

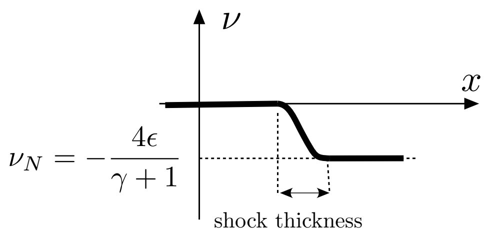
valid up to order $\epsilon^2$

$$\frac{1}{\gamma}\pi + \nu = \mathrm{Pr}\frac{\mathrm{d}\nu}{\mathrm{d}\xi} - 2\epsilon\nu + O(\epsilon^3), \quad \Rightarrow \quad \left(\frac{\gamma+1}{2\gamma}\right)\pi\nu + \mathrm{Pr}\frac{\mathrm{d}\nu}{\mathrm{d}\xi} - 2\epsilon\nu = \frac{\mathrm{d}\pi}{\mathrm{d}\xi} + \frac{\mathrm{d}\nu}{\mathrm{d}\xi} + O(\epsilon^3),$$

$$\pi = -\gamma\nu + O(\epsilon^2), \quad \Rightarrow \quad \boxed{[(\gamma-1) + \mathrm{Pr}]\frac{\mathrm{d}\nu}{\mathrm{d}\xi} = \left(\frac{\gamma+1}{2}\nu + 2\epsilon\right)\nu}$$

Rankine-Hugoniot

$$\nu_N = -4\epsilon\frac{1}{\gamma+1} \qquad \pi_N = -4\epsilon\frac{\gamma}{\gamma+1}$$

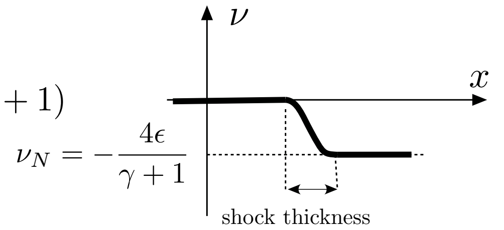


$M_u \equiv \mathcal{D}/a_u > 1$ **Analysis for** $\boxed{\epsilon \equiv M_u - 1 \ll 1}$ (weak shock)

$$\boxed{[(\gamma-1)+\mathrm{Pr}]\frac{\mathrm{d}\nu}{\mathrm{d}\xi} = \left(\frac{\gamma+1}{2}\nu + 2\epsilon\right)\nu}$$

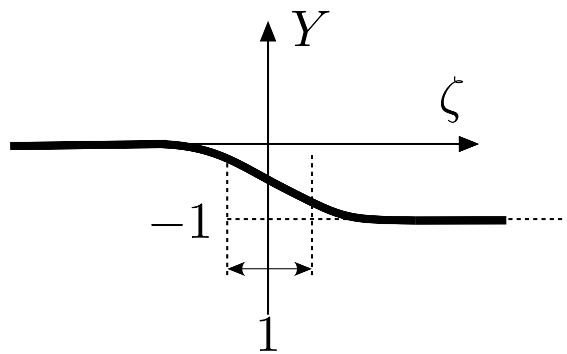
$$\frac{2}{\gamma+1}[(\gamma-1)+\mathrm{Pr}]\frac{\mathrm{d}\nu}{\mathrm{d}\xi} = \nu(\nu-\nu_N) \leqslant 0$$

$\xi = -\infty:$ initial state, $\nu = 0$, $\quad \xi = +\infty:$ shocked gas, $\nu = \nu_N = -4\epsilon/(\gamma+1)$




$$\boxed{Y \equiv \frac{(\gamma+1)}{4\epsilon}\nu \in [0,-1]} \qquad \zeta \equiv \frac{2}{[(\gamma+1)+\mathrm{Pr}]}\epsilon\xi = \frac{2}{[(\gamma+1)+\mathrm{Pr}]}\frac{x}{(\ell/\epsilon)} \qquad \boxed{\zeta = O\left(\frac{x}{\ell/\epsilon}\right)}$$

$$\boxed{\begin{array}{c}\dfrac{\mathrm{d}Y}{\mathrm{d}\zeta} = Y(Y+1) < 0 \\ \zeta = -\infty:\ Y = 0, \qquad \zeta = +\infty:\ Y = -1\end{array}}$$




$$\zeta = \int \frac{\mathrm{d}Y}{Y(Y+1)} \qquad Y(\zeta) = -\frac{\mathrm{e}^{\zeta}}{\mathrm{e}^{\zeta}+1}$$

shock thickness $=$ mean free path$/(M_u - 1)$

microscopic length if $M_u - 1 = O(1)$

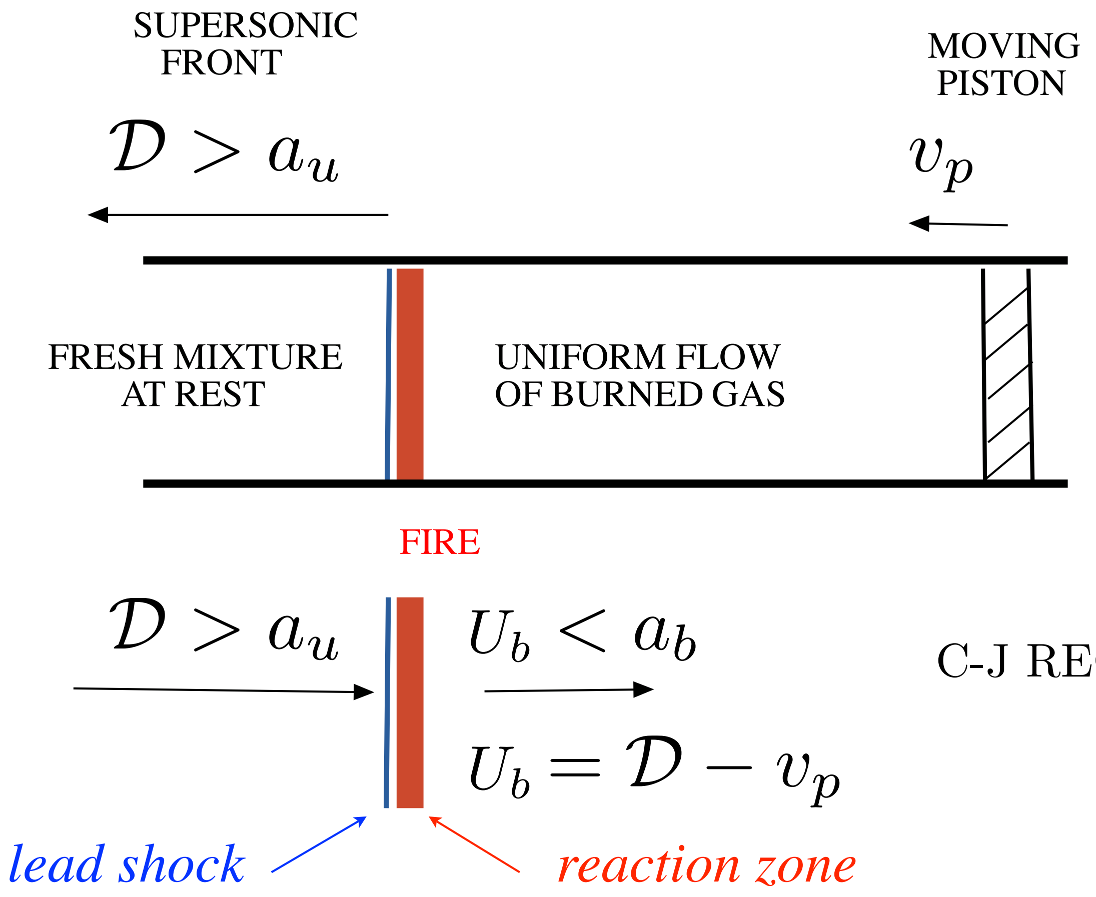
$\boxed{\text{macroscopic length if } (M_u - 1) \ll 1}$

$$\boxed{x = O(l/\epsilon)}$$

# 10-3. Gaseous detonations

**OVERDRIVEN DETONATION**
**REACTING GAS**

**PISTON SUPORTED SUPERSONIC WAVE**




C-J REGIME $U_{b_{CJ}} = a_{b_{CJ}}$

**Abel** 1870, Berthelot et Vielle 1881, Mallard et Le Chatelier 1881, **Mikhel'son** 1893, Chapman 1899, Jouguet 1904,

Vielle 1900, **Zel'dovich** 1940, von Neumann1942, Döring 1943,

# Jump conditions across a planar detonation

## (ZND inner structure of the detonation wave in steady state)

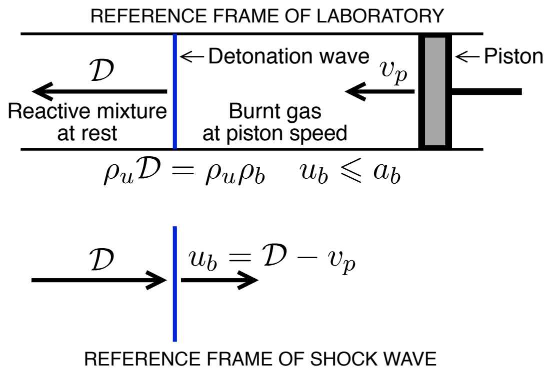
see the historical introduction in the book of ***John H.S Lee*** **« The detonation phenomenon »** *Cambridge University Press* (2008)

**Equality of the fluxes of the conserved scalars (mass, momentum, total energy) on both sides**

**No production terms**

**No chemical kinetics consideration**

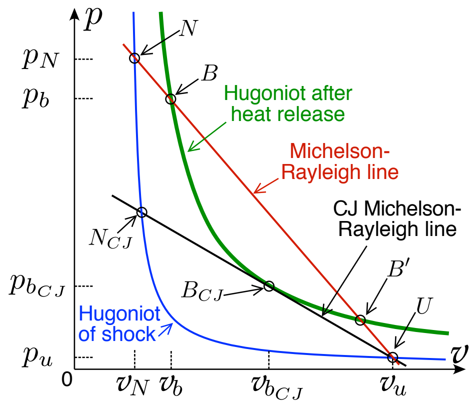
# Mikhelson condition for the CJ detonation (1893)




$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}, \qquad \frac{\partial(\rho u)}{\partial t} = -\frac{\partial}{\partial x}\left(p + \rho u^2 - \mu\frac{\partial u}{\partial x}\right), \qquad \frac{\partial(\rho e_{tot})}{\partial t} = -\frac{\partial}{\partial x}\left[\rho u \boxed{\left(c_p T + \frac{u^2}{2} - q_m \psi\right)} - \lambda\frac{\partial T}{\partial x} - \mu u \frac{\partial u}{\partial x}\right]$$

$$c_p(T_b - T_u) + (u_b^2 - \mathcal{D}^2)/2 = q_m$$

$$\frac{\gamma}{\gamma - 1}\left(\frac{p_b}{\rho_b} - \frac{p_u}{\rho_u}\right) - \frac{1}{2}(p_b - p_u)\left(\frac{1}{\rho_u} + \frac{1}{\rho_b}\right) = q_m$$




$$\mathcal{P} \equiv \frac{(\gamma+1)}{2\gamma}\left(\frac{p}{p_u} - 1\right), \quad \mathcal{V} \equiv \frac{(\gamma+1)}{2}\left(\frac{\rho_u}{\rho} - 1\right) \qquad \boxed{\mathcal{Q} \equiv \frac{\gamma+1}{2}\frac{q_m}{c_p T_u}}$$

$$\boxed{\begin{array}{c}(\mathcal{P}+1)(\mathcal{V}+1) = 1 + \mathcal{Q} \\ \mathcal{P} = -M_u^2 \mathcal{V}\end{array}}$$

quadratic equation for $\mathcal{V}$ $\qquad M_u^2\mathcal{V}^2 + (M_u^2 - 1)\mathcal{V} + \mathcal{Q} = 0$

supersonic combustion wave $\qquad M_u \equiv \mathcal{D}/a_u > 1$

**Lower bound of propagation velocity $\mathcal{D} = \mathcal{D}_{CJ}$**
(called Chapmann-Jouguet 1899 – 1904)

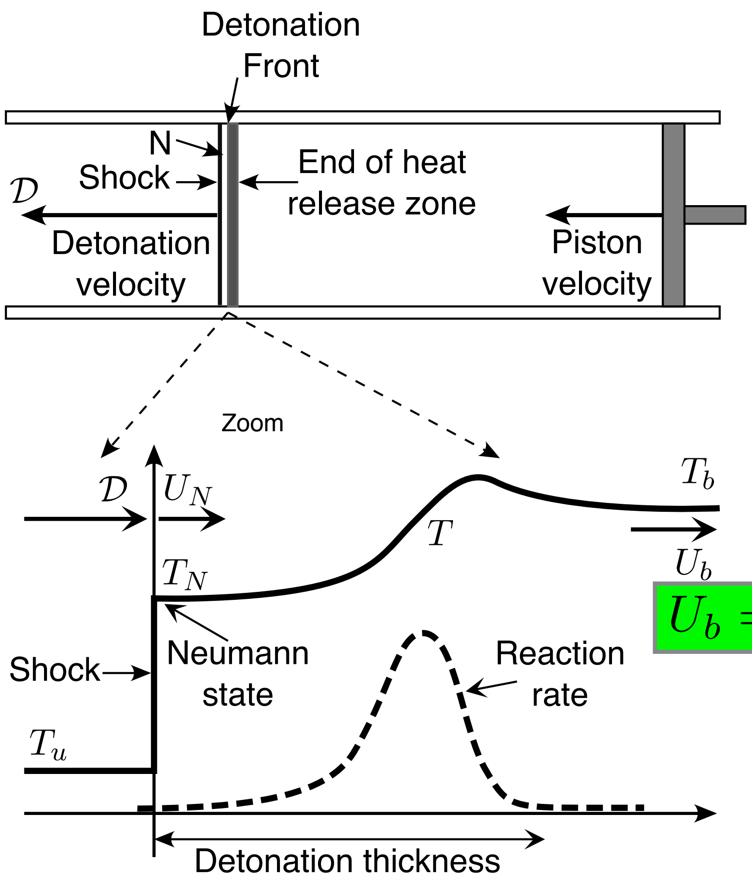
$$(M_u^2 - 1)^2 \geqslant 4\mathcal{Q}M_u^2 \qquad M_u \geqslant M_{u_{CJ}} \equiv \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q}+1}, \qquad \text{Mikhelson (1893)}$$

In the CJ wave the velocity of the burned gas is sonic in the frame of the wave $u_{bCJ} = a_{bCJ}$ (self-sustained wave)
(Rayleigh line is tangent)

In the overdriven detonations $\mathcal{D} > \mathcal{D}_{CJ}$ the velocity of the burned gas is subsonic in the frame of the wave $u_b < a_b$
(piston-supported detonation)

see next slide

# Planar detonation




$U_b = a_b$

thickness ≈ few mms

*Arrhenius law* $\mathrm{e}^{-E/k_BT}$

$$reaction\ rate \approx \frac{\mathrm{e}^{-E/k_BT}}{\tau_{coll}} \qquad \frac{E}{k_BT} \approx 10$$

*elastic collisions*

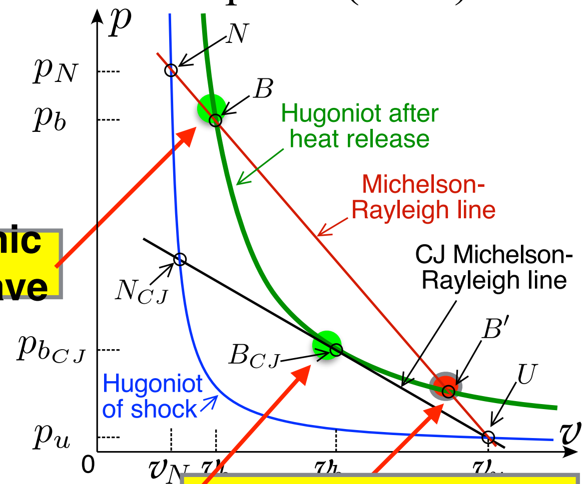
## Overdriven regime / Self sustained wave

## Marginal solution

the so called Chapman-Jouguet wave

Mikhel'son
(1893)

Chapman (1899)




**subsonic flow/wave**

supersonic without shock
non physical
under ordinary conditions

see below

Michelson (1893) Rayleigh (1910)

Sonic condition

C-J detonation

P.Clavin X

Paul Vieille (1900)

# ZND structure of detonations

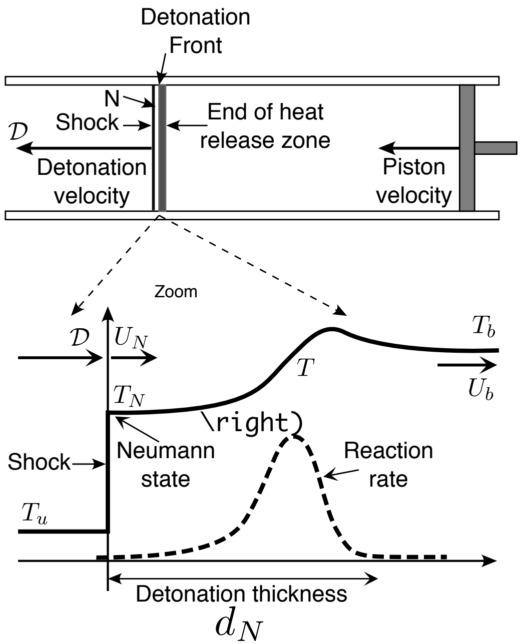
Zeldovich (1940) Neumann (1942) Döring (1944)

Zeldovich

## Detonation = Shock-driven combustion wave

conjectured by Vieille (1900)

structure of the detonation: **inert shock followed by a much larger reaction zone**




## Combustion = large activation energy

***order of magnitude***

$$\frac{E}{k_B T_N} \gg 1 \;\Rightarrow\; \frac{1}{\tau_r(T_N)} \approx \frac{e^{-E/k_B T_N}}{\tau_{coll}} \ll \frac{1}{\tau_{coll}} \qquad \frac{u_N}{a_N} = O(1), \quad d_N \equiv u_N \tau_r(N) \gg a_N \tau_{coll} \approx \ell$$

thickness of the reaction zone $\gg$ thickness of the lead (inert) shock

$$\mathrm{D}_T \approx a_N^2 \tau_{coll}, \quad \frac{D_T}{d_N^2} = \frac{D_T}{(u_N \tau_r)^2} = \left(\frac{a_N}{u_N}\right)^2 \left(\frac{\tau_{coll}}{\tau_r}\right) \frac{1}{\tau_r}, \quad \Rightarrow \quad 1 < \frac{a_N}{u_N} = O(1) \quad \Rightarrow \quad \boxed{\frac{D_T}{d_N^2} \ll \frac{1}{\tau_r}}$$

diffusion rate $\ll$ reaction rate $\Rightarrow$ diffusion terms are negligible in the reaction zone behind the shock

# ZND structure of detonations

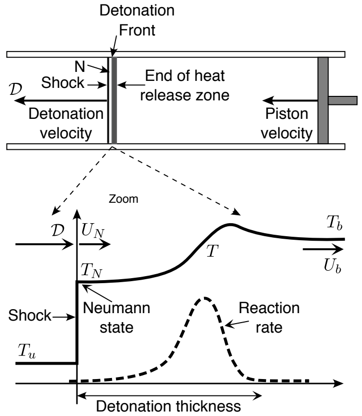
Zeldovich (1940) Neumann (1942) Döring (1944)

Zeldovich

structure of the detonation: **inert shock followed by a much larger reaction zone**

$$\frac{D_T}{d_N^2} \approx \frac{a_N^2 \tau_{coll}}{d_N^2} \approx \frac{\tau_{coll}}{(\tau_r(T_N))^2} \approx \frac{e^{-E/k_B T_N}}{\tau_r(T_N)} \ll \frac{1}{\tau_r(T_N)}$$

diffusion rate $\ll$ reaction rate $\Rightarrow$ **diffusion terms are negligible**




## Formulation

*Reference frame of the lead shock* $(x = 0)$

$$\frac{\mathrm{d}(\rho u)}{\mathrm{d}x} = 0 \qquad \frac{\mathrm{d}p}{\mathrm{d}x} + \rho u \frac{\mathrm{d}u}{\mathrm{d}x} = 0$$

$$\frac{\gamma}{\gamma - 1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{\rho}\right) + u\frac{\mathrm{d}u}{\mathrm{d}x} - q_m \frac{\mathrm{d}\psi}{\mathrm{d}x} = 0 \qquad \rho u \frac{\mathrm{d}\psi}{\mathrm{d}x} = \rho \frac{\dot{w}(T,\psi)}{\tau_r(T_N)}$$

$$\psi \in [0,1] \qquad \begin{array}{l} \dot{w}(T,\psi = 1) = 1 \\ \dot{w}(T,\psi = 0) = 0. \end{array}$$

$$x = 0: \quad u = u_N, \quad \rho = \rho_N, \quad p = p_N, \quad \psi = 1, \quad \dot{w} = 1$$
$$x \to \infty: \quad u = u_b, \quad \rho = \rho_b, \quad p = p_b, \quad \psi = 0, \quad \dot{w} = 0$$

detonation thickness $d_N = u_N \tau_r(T_N)$

$$a^2 = \gamma \frac{p}{\rho}$$ *Elimination of* $p$ *and* $\rho$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{\rho}\right) = p\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{p}{\rho u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} = \frac{a^2}{\gamma u}\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}u}{\mathrm{d}x} \quad \Rightarrow \quad \frac{\gamma}{\gamma - 1}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p}{\rho}\right) + u\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{(\gamma - 1)u}(a^2 - u^2)\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$(a^2 - u^2)\frac{\mathrm{d}u}{\mathrm{d}x} = (\gamma - 1) q_m u \frac{\mathrm{d}\psi}{\mathrm{d}x}, \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}u^2}{\mathrm{d}\psi} = 2(\gamma - 1) q_m \frac{u^2}{(a^2 - u^2)}}$$

*Phase portrait in the plan $\psi$ - $u^2$*

$$\boxed{\frac{du^2}{d\psi} = 2(\gamma-1)q_m \frac{u^2}{(a^2-u^2)}}$$

Initial state $\psi = 0: \quad u^2 = \mathcal{D}^2,\ a^2 = a_u^2$

Neumann state $\psi = 0: \quad u^2 = u_N^2,\ a^2 = a_N^2$

energy eq.

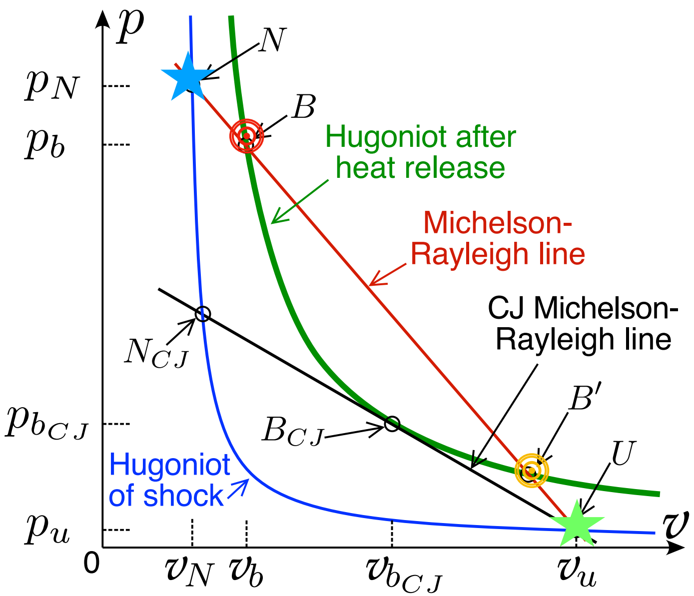
$$\underbrace{\frac{1}{\gamma-1}a^2}_{C_pT} + \frac{1}{2}u^2 - q_m\psi = \frac{1}{\gamma-1}a_u^2 + \frac{1}{2}\mathcal{D}^2$$

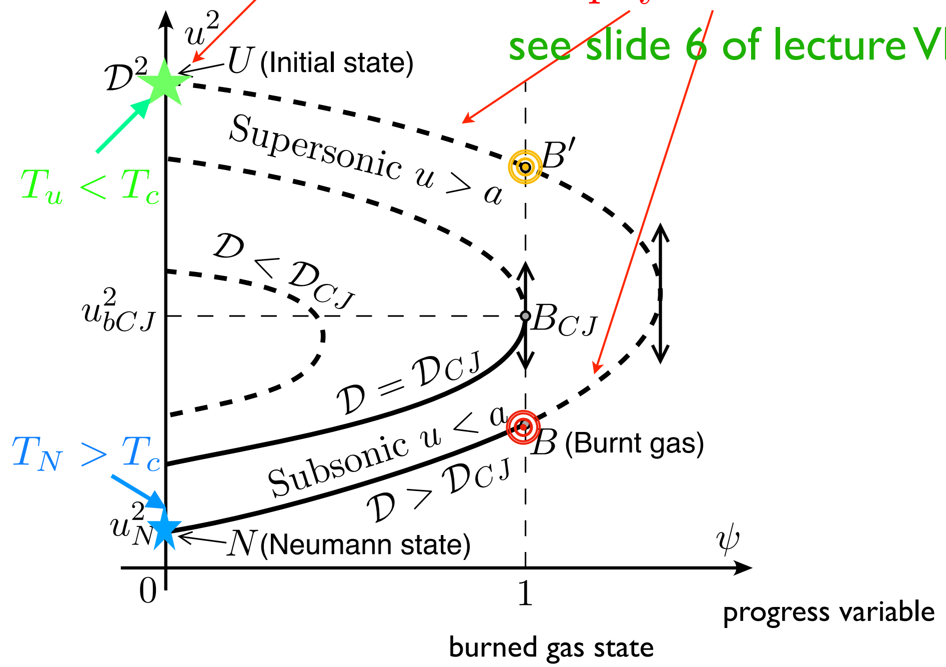
zero reaction rate

no supersonic wave without a leading shock

Non physical branch

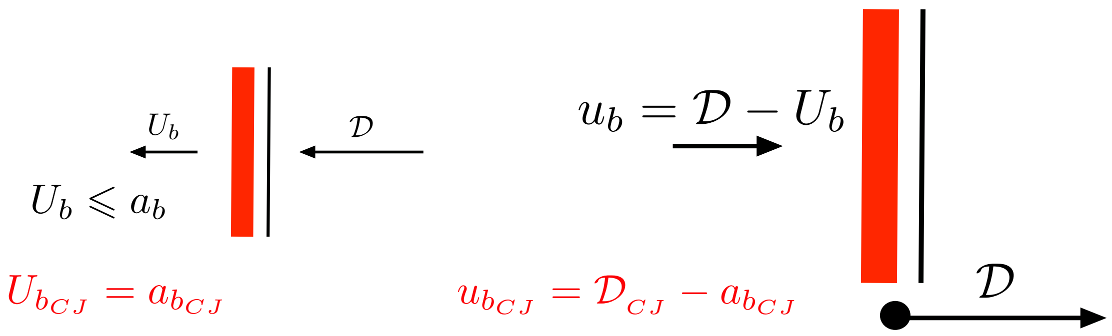
see slide 6 of lecture VI



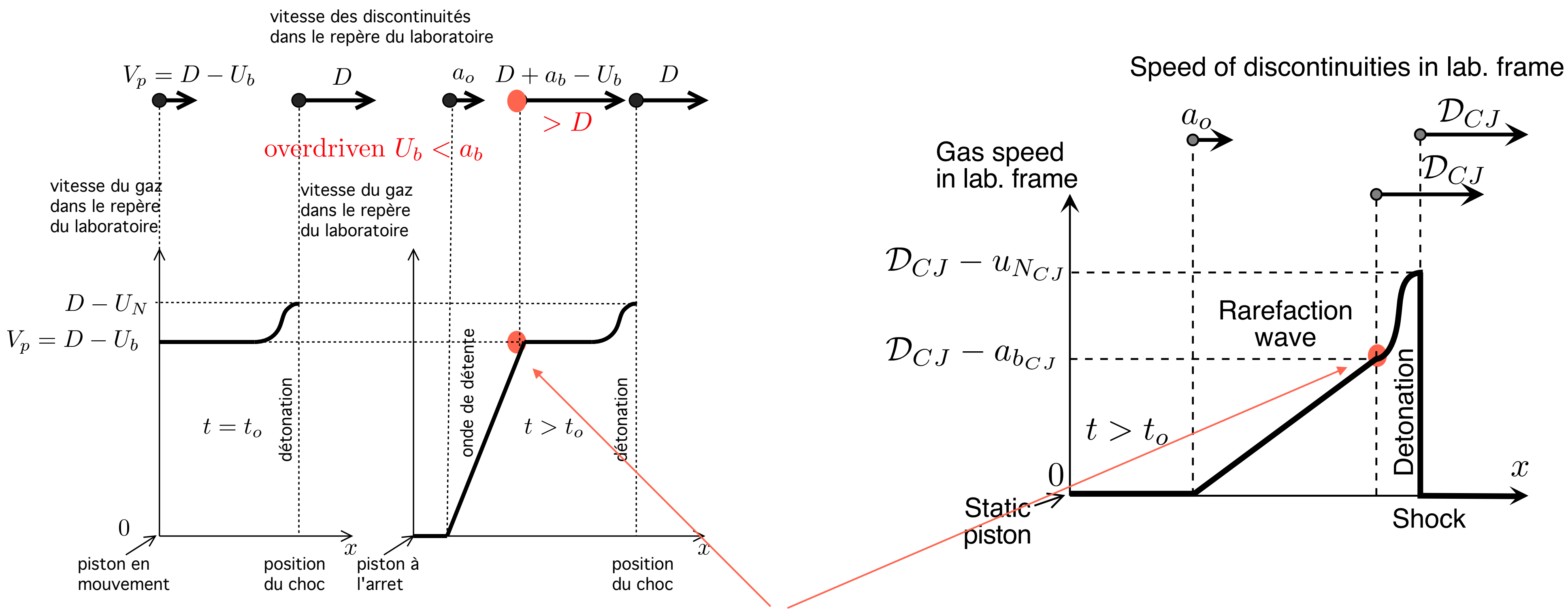

OK with the Vieille's conjecture

# C-J Detonation = self-propagating wave



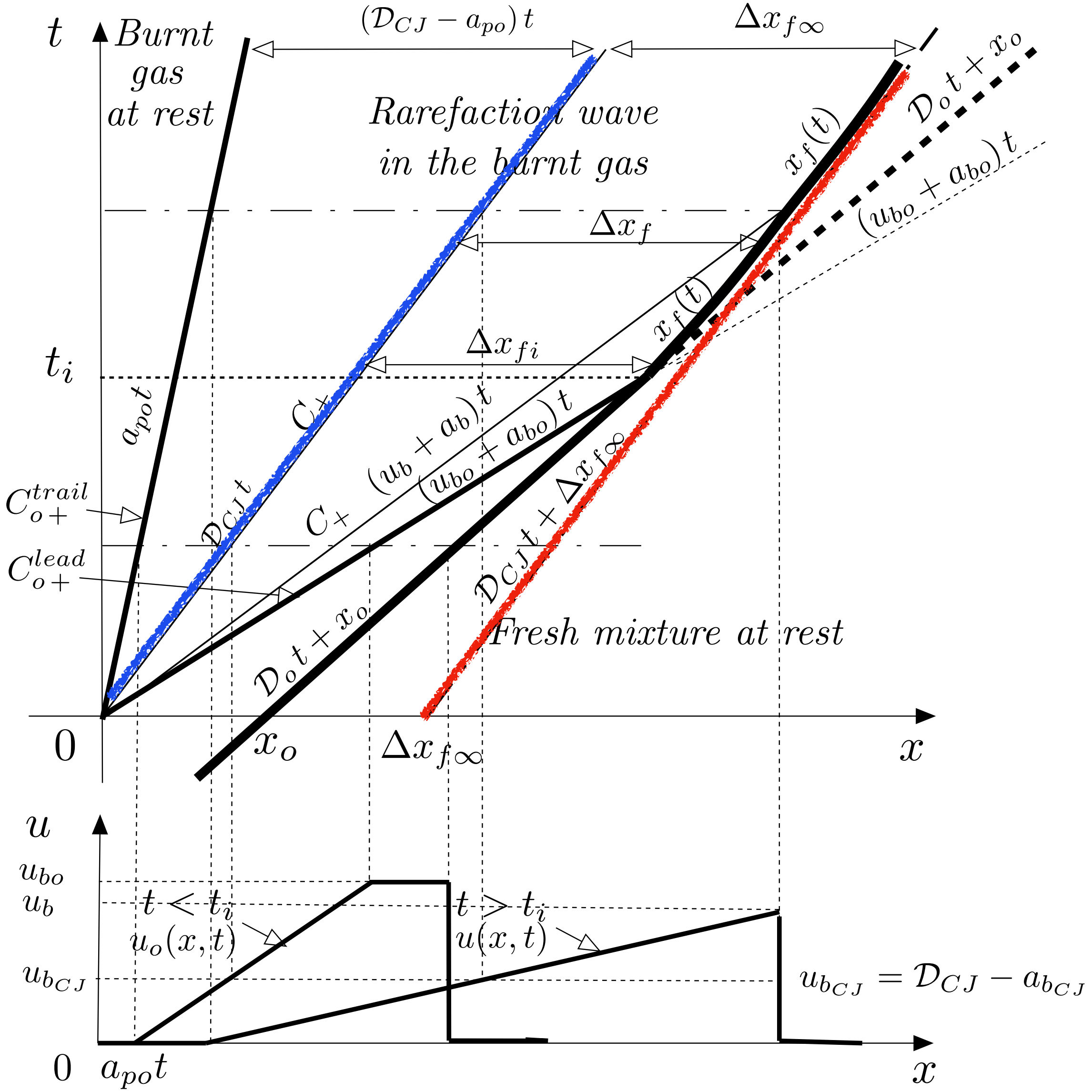

## 10-4. Selection mechanism of the CJ wave

Rarefaction wave in the burnt gas when the piston is suddenly stopped




leading edge of the rarefaction wave = speed of sound /gas flow

Clavin, Denet *JFM* (2018) **845,** 170 – 202




$$\sqrt{\frac{\mathrm{d}x_f}{\mathrm{d}t} - \mathcal{D}_{CJ}} \;\propto\; \frac{x_f}{t}$$

$$\frac{\mathcal{D}_{CJ}}{x_f(t) - \mathcal{D}_{CJ}t} \propto \frac{1}{t} + \text{constant}$$

$$\lim_{t\to\infty} x_f(t) = \mathcal{D}_{CJ}t + \text{constant}$$

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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## Lecture XI
## **Initiation of detonations**
## Part 1: Direct Initiation

# Lectures 11: Initiation of detonation

## Lecture 11-a: Direct initiation

*Background*
*Rarefaction wave behind a CJ detonation*
*Critical energy*
*Critical dynamics*

## Lecture 11-b: Spontaneous initiation and quenching

*Initiation at high temperature*
*Spontaneous quenching*

## Lecture 11-c: Deflagration to Detonation Transition

*Basic ingredients*
*Experiments*
*Runaway phenomenon*
*Intrinsic DDT mechanism of laminar flames*

# Lectures 11: Initiation of detonation

## Lecture 11-a: Direct initiation
*Background*

### *What is the direct initiation of detonation ?*

Formation of a detonation in open space produced by the rapid deposition of a powerful concentrated energy source

### *Detonable mixtures*

Mixtures in which self-sustained planar detonations can propagate. The Neumann temperature (just behind the lead shock) of the CJ detonation should be larger than the crossover temperature:

$$\boxed{T_{N_{CJ}} > T_c \,:}$$

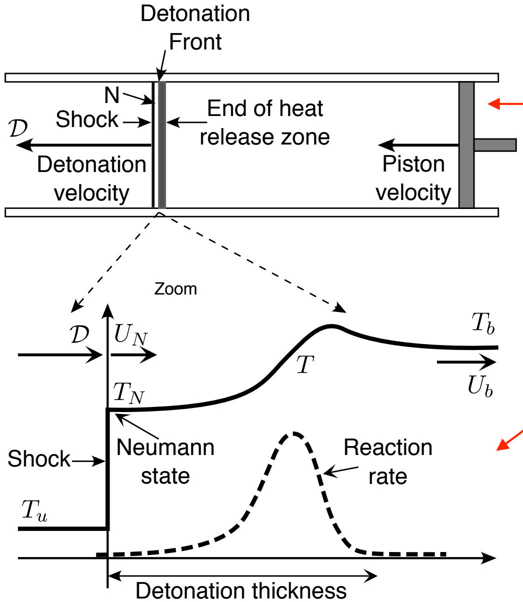
composition

crossover temperature
chemical kinetics

$$(\gamma - 1) M^2_{u_{CJ}} > 1 \Rightarrow T_{N_{CJ}} \propto q_m / c_p \qquad 850\,K < T_c < 1250\,K \,:$$

heat release per unit mass of the deficient species (fuel or oxygen in lean and rich mixtures respectively)

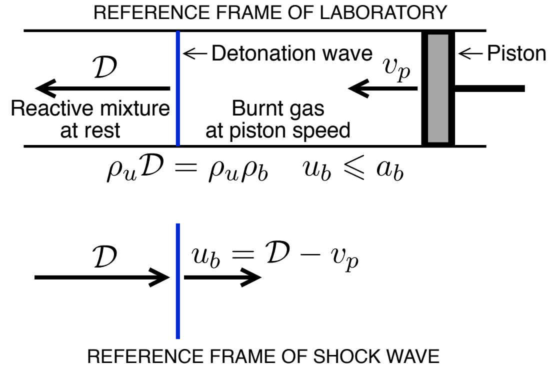
# *Planar detonations in steady state. Self-sustained regime: CJ wave* *(reminder)*



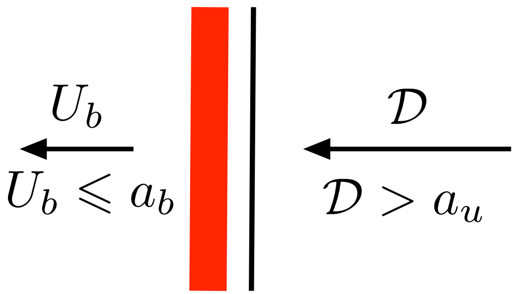

flow velocity in the laboratory frame

reference frame of the lead shock wave

$$\rho_u \mathcal{D} = \rho_b U_b \qquad u_b = \left(1 - \frac{\rho_u}{\rho_b}\right)\mathcal{D}$$



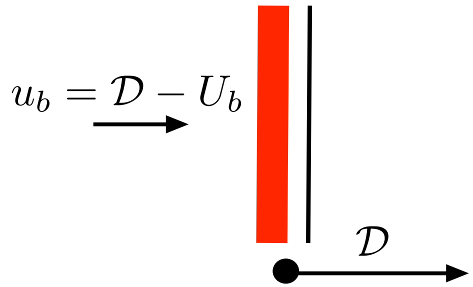

Conservation of mass, momentum and energy across the wave lead to a quadratic equation for




$$\mathcal{V} \equiv \frac{(\gamma+1)}{2}\left(\frac{\rho_u}{\rho_b} - 1\right) = -\frac{(\gamma+1)}{2}\frac{u_b}{\mathcal{D}}$$

$$\boxed{M_u^2 \mathcal{V}^2 + (M_u^2 - 1)\mathcal{V} + Q = 0}$$

$$Q \equiv \frac{\gamma+1}{2}\frac{q_m}{c_p T_u} \qquad M_u \equiv \mathcal{D}/a_u$$




laboratory frame

## CJ detonation

**Lower bound of propagation velocity $\mathcal{D} = \mathcal{D}_{CJ}$**

**marginal solution**

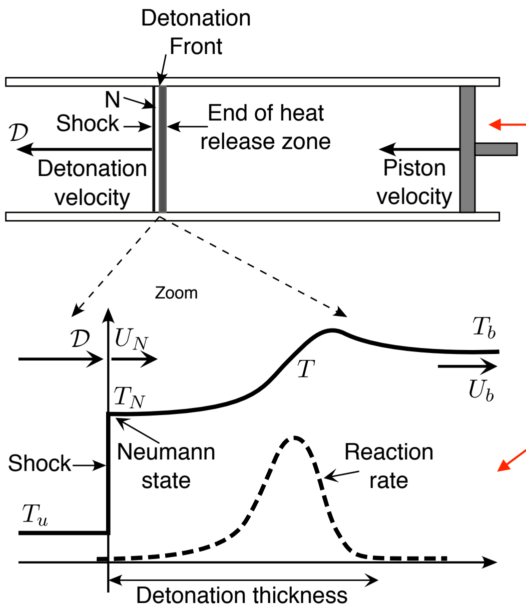
## sonic condition

$$U_{b_{CJ}} = a_{b_{CJ}} \qquad u_{b_{CJ}} = \mathcal{D}_{CJ} - a_{b_{CJ}}$$

$$\mathcal{V}_{CJ} = -\frac{(M_{u_{CJ}}^2 - 1)}{2M_{u_{CJ}}^2} \qquad M_{u_{CJ}}^2 \gg 1 \quad \Rightarrow \quad \mathcal{V}_{CJ} \approx -1/2, \;\; u_{b_{CJ}} \approx \mathcal{D}_{CJ}/(\gamma+1)$$

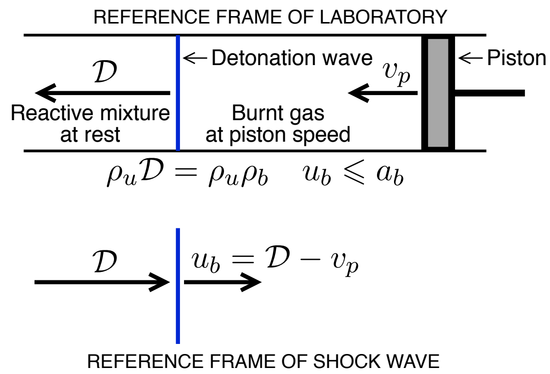
Lecture 11-a: **Direct initiation**

*Rarefaction wave behind a CJ detonation*




flow velocity in the laboratory frame

reference frame of the lead shock wave




CJ DETONATION: SELF PROPAGATING

Stop the piston $\Rightarrow$ Rarefaction wave

Flow of burned gas behind a CJ detonation viewed as a discontinuity

**self-similar form** Zeldovich (1942) Taylor (1950)

$$u(r,t) = v(\mathrm{x}) \qquad \boxed{\mathrm{x} \equiv r/t}$$

flow velocity in the laboratory frame

# Rarefaction wave behind a planar CJ detonation

(discontinuous model)

**self-similar form**

$$\boxed{u(r,t)=v(\mathrm{x})} \qquad \boxed{\mathrm{x}\equiv r/t}$$

flow velocity in the laboratory frame

$$\boxed{\frac{\partial}{\partial r}=\frac{1}{t}\frac{\mathrm{d}}{\mathrm{dx}}, \qquad \frac{\partial}{\partial t}=-\frac{r}{t^2}\frac{\mathrm{d}}{\mathrm{dx}}=-\frac{\mathrm{x}}{t}\frac{\mathrm{d}}{\mathrm{dx}}}$$

Euler eqs. in a planar geometry $\Rightarrow$

$$\begin{cases} (v-\mathrm{x})\dfrac{1}{\rho}\dfrac{\mathrm{d}\rho}{\mathrm{dx}}+\dfrac{\mathrm{d}v}{\mathrm{dx}}=0 \\ a^2\dfrac{1}{\rho}\dfrac{\mathrm{d}\rho}{\mathrm{dx}}+(v-\mathrm{x})\dfrac{\mathrm{d}v}{\mathrm{dx}}=0 \end{cases}$$

$\rho(\mathrm{x})$ $\quad a(\mathrm{x})$

2 ordinary diff. eqs of 1st order $v(\mathrm{x})$ $\quad \rho(\mathrm{x})$

$$\Downarrow$$

$$v=\frac{r}{t}-a \qquad v=\mathrm{x}-a \qquad \Leftarrow \qquad \left[1-\left(\frac{v-\mathrm{x}}{a}\right)^2\right]\frac{\mathrm{d}v}{\mathrm{dx}}=0$$

Riemann invariant $J_-$

$$\frac{2a}{\gamma-1}-v=\frac{2a_{b_{CJ}}}{\gamma-1}-v_{b_{CJ}} \Rightarrow a=\frac{(\gamma-1)}{2}(v-v_{b_{CJ}})+a_{b_{CJ}}$$

elimination of $a$

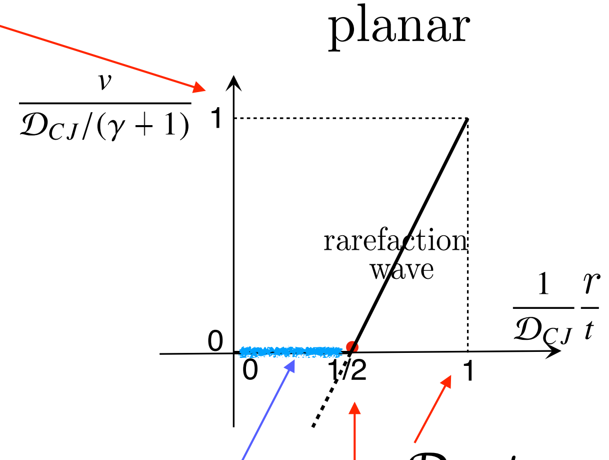
$$\frac{(\gamma+1)}{2}(v-v_{b_{CJ}})=\mathrm{x}-(a_{b_{CJ}}+v_{b_{CJ}})$$

rarefaction wave = straight line

sonic condition

$$a_{b_{CJ}}+v_{b_{CJ}}=\mathcal{D}_{CJ} \Rightarrow \boxed{(\gamma+1)\frac{v-v_{bCJ}}{\mathcal{D}_{CJ}}=2\left[\frac{\mathrm{x}}{\mathcal{D}_{CJ}}-1\right]}$$

$$u_{b_{CJ}}\approx\mathcal{D}_{CJ}/(\gamma+1)$$




$r_{front_{CJ}}=\mathcal{D}_{CJ}t$

core of burned gas at rest

weak discontinuity: sonic velocity

$\mathrm{x}=a_b, \quad r=a_b\,t$

$M^2_{u_{CJ}}\gg 1$ (for simplicity) $\quad a_{b_{CJ}}\approx\dfrac{\gamma}{\gamma+1}\mathcal{D}_{CJ} \quad v_{b_{CJ}}=\dfrac{\mathcal{D}_{CJ}}{\gamma+1}$

$$v=0:\ a_b=-\frac{\gamma-1}{2}v_{b_{CJ}}+a_{b_{CJ}}\approx\frac{\mathcal{D}_{CJ}}{2}$$

$$v=0:\ \mathrm{x}=-\frac{\gamma+1}{2}v_{b_{CJ}}+\mathcal{D}_{b_{CJ}}\approx\frac{\mathcal{D}_{b_{CJ}}}{2}$$

$v(\mathrm{x})$ is a straight line

P.Clavin XI **Rarefaction** wave behind a **spherical CJ** detonation

*(discontinuous model)*

$M^2_{u_{CJ}} \gg 1$ for simplicity

self-similar solution for the flow of burned gas=rarefaction wave

$v(\mathrm{x})$ $\boxed{\mathrm{x} \equiv r/t}$

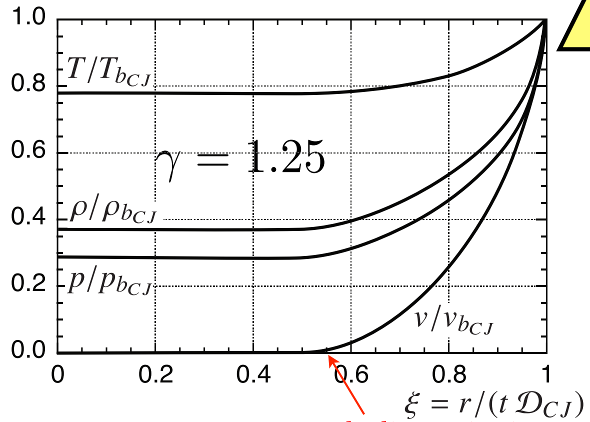
flow velocity in the laboratory frame

$$\frac{\partial}{\partial r} = \frac{1}{t}\frac{\mathrm{d}}{\mathrm{dx}}, \qquad \frac{\partial}{\partial t} = -\frac{r}{t^2}\frac{\mathrm{d}}{\mathrm{dx}} = -\frac{\mathrm{x}}{t}\frac{\mathrm{d}}{\mathrm{dx}}$$

spherical geometry

$$\left.\begin{aligned} \frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{u}{\rho}\frac{\partial \rho}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) = 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial r} \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} (v-\mathrm{x})\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{dx}} + \frac{\mathrm{d}v}{\mathrm{dx}} + \boxed{\frac{2v}{\mathrm{x}}} = 0 \\ a^2\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{dx}} + (v-\mathrm{x})\frac{\mathrm{d}v}{\mathrm{dx}} = 0 \end{aligned}\right. \Rightarrow \frac{\mathrm{x}}{v}\frac{\mathrm{d}v}{\mathrm{dx}} = \frac{2}{\left[1 - \left(\frac{v-\mathrm{x}}{a}\right)^2\right]}$$

spherical




CJ condition

weak discontinuity

$\boxed{r = a_o t}$

*self-similar solution of the first kind* (Zeldovich-Barenblatt 1958)

$$\xi \equiv r/(\mathcal{D}_{CJ}t), \quad v = v_{b_{CJ}}\mathcal{U}(\xi), \quad \rho = \rho_{b_{CJ}}\mathcal{R}(\xi)$$

$$v_{b_{CJ}} = \mathcal{D}_{CJ} - a_{b_{CJ}}$$

$M^2_{u_{CJ}} \gg 1$ strong shock approximation: $\mathcal{D}_{b_{CJ}}/v_{b_{CJ}} = \gamma + 1, \quad a/v_{b_{CJ}} = \gamma \mathcal{R}^{(\gamma-1)/2}$

$$\boxed{\begin{gathered}[\mathcal{U} - (\gamma+1)\xi]\frac{1}{\mathcal{R}}\frac{\mathrm{d}\mathcal{R}}{\mathrm{d}\xi} + \frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\xi} + \frac{2\mathcal{U}}{\xi} = 0 \qquad \gamma^2\mathcal{R}^{\gamma-1}\frac{1}{\mathcal{R}}\frac{\mathrm{d}R}{\mathrm{d}\xi} + [\mathcal{U} - (\gamma+1)\xi]\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}\xi} = 0 \\ \xi = 1: \ \mathcal{U} = 1, \ \mathcal{R} = 1.\end{gathered}}$$

CJ condition

$\mathrm{d}\mathcal{U}/\mathrm{d}\xi$ diverges at $\xi = 1$

$$\lim_{\xi\to 1}(1-\mathcal{U})^2 = \frac{2\gamma}{\gamma+2}(1-\xi)$$

start the numerical integration at $\xi = 1$ : $\mathcal{U} = 1, \ \dfrac{d\mathcal{U}}{\mathrm{d}\xi} = \sqrt{\dfrac{\gamma}{2(\gamma+2)}}\dfrac{1}{(1-\gamma)}$

stop the calculation at $\xi_o$ at which $\mathcal{U} = 0$; uniform solution in $0 \leqslant \xi \leqslant \xi_o$

spherical kernel of burnt gas at rest whose radius increasing linearly with time

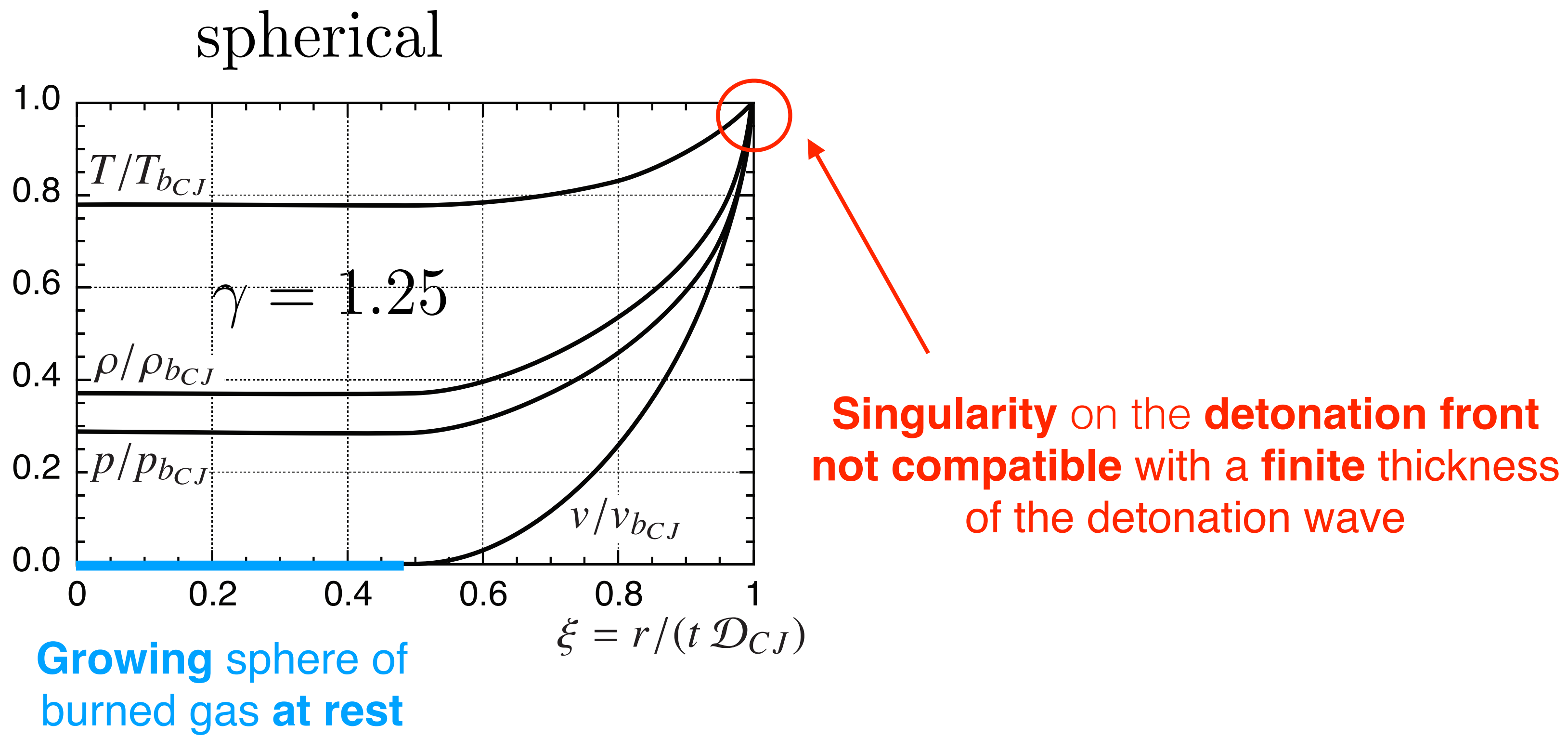
$$\xi \equiv \frac{r}{\mathcal{D}_{CJ}\, t}$$

# ***Self-similar*** *solution*

## *Rarefaction wave behind a spherical CJ detonation*

***discontinuous*** *model (no other length scale than the radius)*

Zeldovich (1942)-Taylor (1950)



# *Direct initiation ?*

Initiation by releasing quasi instantaneously an amont of energy in a quasi point. (e.g. explosive charge)

Successful initiation occur above a critical energy

$$E > E_c$$

At **early time** after deposition, the size of the spherical wave is very small and the energy liberated by the exothermal reaction is **negligible** in front of the energy that has been deposited.

Therefore, the initial condition for the study of direct initiation is a **point blast wave** which is described by a spherical **self-similar** solution of **inert** Euler equations (no time and space scale)

# ***Point blast wave explosion*** *in an* ***inert gas*** *(spherical geometry)*

(Taylor 1941 Sedov 1946)

The shock velocity $\mathcal{D}(t)$ varying with the time $\Rightarrow$ the entropy jump across the shock is not constant however the dissipation can be neglected outside the shock.

The flow is solution of the Euler equations completed by the entropy wave equation

$$\text{Euler eqs.} \; + \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\left(\frac{p}{\rho^\gamma}\right) = 0$$

and satisfies to the Rankine-Hugoniot condition on the lead shock treated as a discontinuity:

Strong shock $\quad M_u \equiv \mathcal{D}/a_u \gg 1 \quad \Rightarrow \quad v_N = \dfrac{2}{\gamma+1}\mathcal{D}(t), \quad \rho_N = \dfrac{\gamma+1}{\gamma-1}\rho_u, \quad p_N = \dfrac{2}{\gamma+1}\rho_u\mathcal{D}^2(t) \quad$ where $\quad \mathcal{D}(t) \equiv \dfrac{\mathrm{d}r_f}{\mathrm{d}t}$

look for a self similar solution in the form $\quad \xi \equiv \dfrac{r}{r_f(t)}, \quad v = \mathcal{D}(t)\mathcal{V}(\xi), \quad \rho = \rho_u\mathcal{R}(\xi), \quad p = \rho_u\mathcal{D}(t)^2\mathcal{P}(\xi)$

$\Rightarrow$ 3 o.d.e. for $\mathcal{V}(\xi),\ \mathcal{R}(\xi),\ \mathcal{P}(\xi)$ with $\quad \xi = 1: \quad \mathcal{V} = 2/(\gamma+1),\ \mathcal{R} = (\gamma+1)/(\gamma-1),\ \mathcal{P} = 2/\gamma+1,$

The trajectory of the lead shock $r = r_f(t)$ is obtained by the following dimensional analysis:

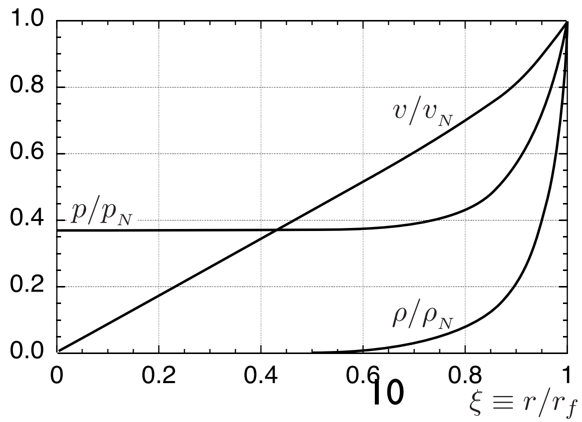
2 dimensional parameters $E$ and $\rho_u$ $\Rightarrow$ a single non-dimensional parameter can be built with $r$ and $t$ : $r(\rho_u/Et^2)^{1/5}$

$$\boxed{r_f(t) = b(\gamma)\left(\frac{E}{\rho_u}\right)^{1/5} t^{2/5}} \Rightarrow \boxed{\mathcal{D}(t) \equiv \dot{r}_f(t) = \frac{2\,b(\gamma)}{5}\left(\frac{E}{\rho_u}\right)^{1/5} t^{-3/5}} \qquad \boxed{\rho_u\mathcal{D}^2 r^3 \approx (2/5)^2 E}$$

conservation of energy : $\quad 4\pi\displaystyle\int_0^{r_f(t)} \rho\left[\frac{1}{\gamma-1}\frac{p}{\rho} + \frac{v^2}{2}\right] r^2\mathrm{d}r = E \quad \Rightarrow \quad b = 1.0033..$ for $\gamma = 1.4$

$$\text{dimension of } E = \frac{\text{mass} \times \text{length}^2}{\text{time}^2}$$
$$\text{dimension of } \rho = \frac{\text{mass}}{\text{length}^3}$$

Flow field
of the Taylor-Sedov blast wave



# *Point blast wave **explosion** in a combustible gas*

## Successful direct initiation: Transition between 2 self-similar solutions

$$E > E_c : \left\{ \begin{array}{l} \text{point blast explosion} \\ \quad \text{Sedov-Taylor} \searrow \text{spherical CJ wave} \\ \qquad\qquad\qquad\quad \text{Zeldovich-Taylor} \end{array} \right. \qquad \left\{ D_f(t) \searrow \mathcal{D}_{CJ} \right.$$

$$r_f(t) \nearrow \infty$$

$$E < E_c : \left\{ \begin{array}{l} \text{point blast explosion} \\ \qquad \searrow \text{fluid at rest } v = 0 \end{array} \right. \qquad \left\{ D_f(t) \searrow a_u \right.$$

1st numerical analysis without modification of the **inner structure** of the lead detonation

(Detonation = discontinuity) $\Rightarrow$ no criticality !
no length scale — no ignition failure

(Korobeinikov 1971, Liñan et al. 2012)

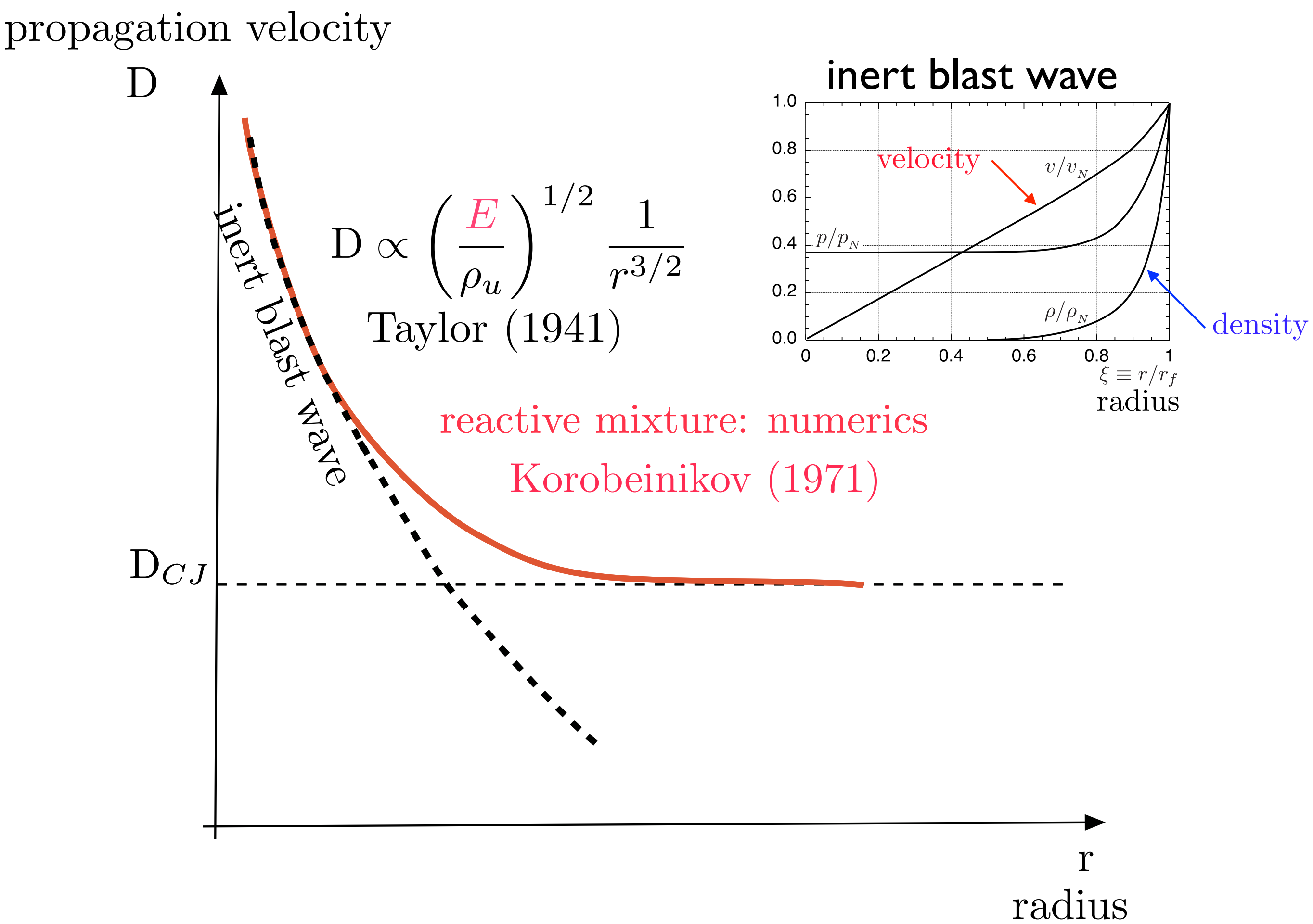
$D_f(t) \searrow \mathcal{D}_{CJ}$ $\boxed{\forall\, E}$ at $r_f \approx r_f^*$ and $t \approx t^*$ corresponding to $\rho_u r_f^{*3} \mathcal{D}_{CJ}^2 \approx E,\ t^* \approx r_f^*/\mathcal{D}_{CJ}$

# First numerics in a spherical geometry

Korobeinikov (1971)
Detonation = discontinuity

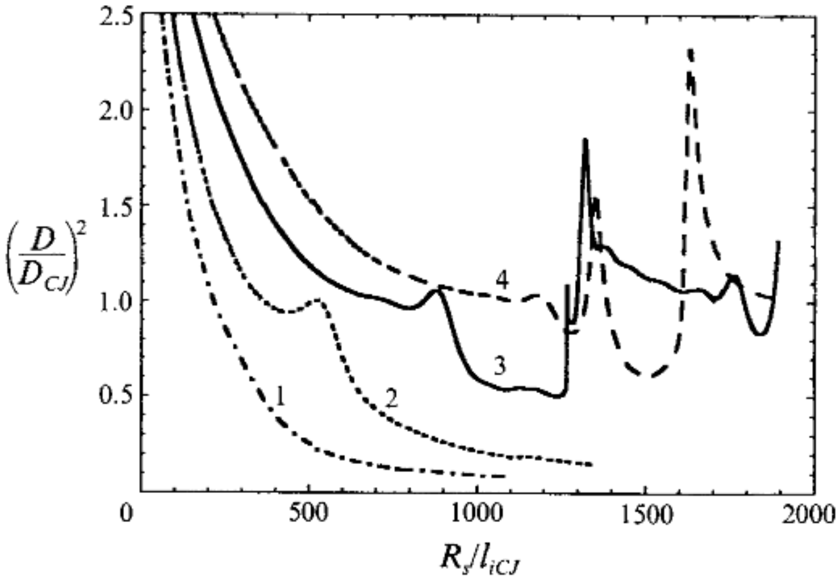
**( zero detonation thickness:)**

**No critical energy !**




**Conclusion: the critical energy is due to modifications of the inner structure of the detonation**

Direct numerical simulations of the flow in spherical geometry including the **unsteady** inner structure of the detonation show that there is a critical energy for a successful initiation; below, the initiation fails and the shock velocity decreases to the sound speed in an inert mixture




*L. He, P. Clavin JFM (1994)* ***277*** *p. 227-248*

*C.A. Eckett et al. JFM (2000)* ***421*** *p. 147-183*

## Critical energy: Zeldovich criterion (1965)

### Order of magnitude estimate for the critical energy by dimensional analysis

the time of the blast wave velocity to reach $\mathcal{D}_{CJ}$ = reaction time at the Neumann state of the CJ wave

$$\text{criticality}: \tau^*_{CJ} = \tau_{N_{CJ}}$$

$\mathcal{D}(t) \approx (E/\rho_u)^{1/5}/t^{3/5}$

self-similar blast wave

$\mathcal{D}/u_N = (\gamma+1)/\gamma-1)$

strong shock

$$\tau^*_{CJ} \approx (E/\rho_u)^{1/3}(\mathcal{D}_{CJ})^{-5/3} = d_{CJ}/u_{N_{CJ}} \approx \frac{\gamma+1}{\gamma-1} d_{CJ}(\mathcal{D}_{CJ})^{-1}$$

$$(E_c/\rho_u)^{1/3} \approx \frac{\gamma+1}{\gamma-1} d_{CJ}\mathcal{D}^{2/3}_{CJ}$$

strong CJ wave

$\mathcal{D}^2_{CJ} \approx 2(\gamma^2-1)q_m$

$$\boxed{E_c \approx 2\rho_u q_m \frac{(\gamma+1)^4}{(\gamma-1)^2} d^3_{CJ}}$$

Smaller by many orders of magnitude than in experiments ! $10^{-5}$ - $10^{-6}$

Lee (1984)

Nonlinear curvature induced modification to the inner structure and fully unsteady effects are essential for a correct estimation of the critical energy

Lecture 11-a: **Direct initiation**

*Critical energy*

*Nonlinear **curvature** effect in spherical geometry*

He Clavin *JFM (1994)* **277** *p. 227-248*

**Quasi-steady** analysis for **large** activation energy

Non linear curvature effect steady state approximation of the spherical detonation structure

**Turning point in the parameter space «radius-velocity »:**
**there is no spherical CJ detonation below a critical radius**

Generic equation of a turning point: $\Theta \mathrm{e}^{-\Theta} = K$

unknown solution: $\Theta$ parameter K

Semenov equation for thermal explosion & Thermal quenching of flames….

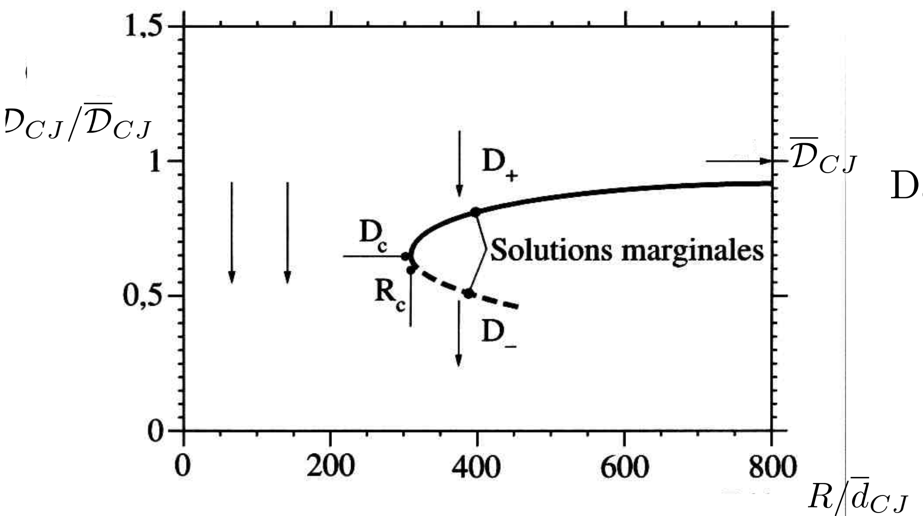
$$\Theta \equiv 2\beta_N \left( \frac{\overline{\mathcal{D}}_{CJ} - \mathcal{D}_{CJ}}{\overline{\mathcal{D}}_{CJ}} \right) \qquad K \equiv \frac{16\gamma^2}{\gamma^2 - 1} \beta_N \frac{\overline{d}_{CJ}}{R}$$

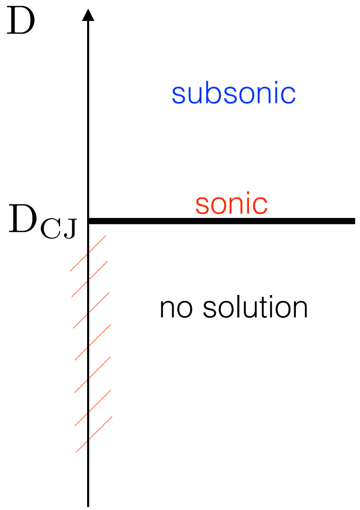
$K > K^* = 1/\mathrm{e}$ : no solution

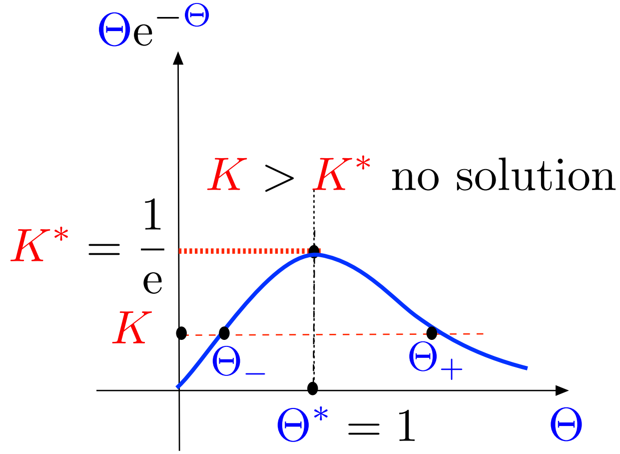
$K < K^* = 1/\mathrm{e}$ : two solutions

$$\Theta_- < \Theta^* = 1 < \Theta_+$$

collapsing for $K = K^* : \Theta_\pm = \Theta^* = 1$




$$R < R_c$$

**No spherical CJ : overdriven regimes that are damped by the rarefaction wave!**

$$K^* = 1/\mathrm{e} \quad \Rightarrow \quad R_c/\overline{d}_{CJ} \approx 10^2$$ OK with DNS

# Details of the asymptotic analysis

*He Clavin JFM (1994) **277** p. 227-248*

*Nonlinear curvature effect of a spherical CJ detonation* $\nabla.\mathbf{j} = \frac{1}{r^2}\frac{\partial(r^2 j)}{\partial r} = \frac{\partial j}{\partial r} + \frac{2}{r}j$

reference frame of the lead shock $x = r_f(t) - r \quad u = \mathcal{D} - v \quad \mathrm{d}r_f(t)/\mathrm{d}t = \mathcal{D} \quad \partial/\partial r \to -\partial/\partial x \quad \partial/\partial t \to \partial/\partial t + \mathcal{D}\partial/\partial x$

Euler eqs.

mass $$\boxed{\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial(\rho u)}{\partial x} + \frac{2}{r_f - \cancel{x}}\rho(\mathcal{D} - u) = 0}$$

momentum $$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}$$

energy $$\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right](c_pT - q_m\psi) - \frac{1}{\rho}\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right]p = 0$$

$\times\rho \Rightarrow$

$$\cancel{\frac{\partial(\rho u)}{\partial t}} + \frac{\partial(\rho u^2 + p)}{\partial x} + \frac{2}{r_f - \cancel{x}}\rho(\mathcal{D} - u)u = \cancel{\rho\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}}$$

$\Rightarrow$

$$\left[\cancel{\frac{\partial}{\partial t}} + u\frac{\partial}{\partial x}\right]\left(\underbrace{c_pT}_{\frac{\gamma}{\gamma-1}\frac{p}{\rho}} + \frac{u^2}{2} - q_m\psi\right) - \cancel{\frac{1}{\rho}\frac{\partial p}{\partial t}} - u\cancel{\frac{\mathrm{d}\mathcal{D}}{\mathrm{d}t}} = 0$$

Quasi-steady state approximation Large radius $\boxed{\epsilon \equiv d_{CJ}/r_f \ll 1}$

detonation thickness curved detonation

Integration across the inner structure $x = 0:$ Neumann state, $\rho_N u_N = \rho_u \mathcal{D}$, $\quad x = d:$ bunrt gas

First order approximation

unperturbed planar solution : $\overline{\rho}(x)\overline{u}(x) = \rho_u\mathcal{D}$

$$\boxed{\begin{aligned} &\frac{(\rho_b u_b - \rho_u\mathcal{D})}{\rho_u\mathcal{D}} \approx -I_1 \\ &\frac{(\rho_b u_b^2 + p_b) - (\rho_u\mathcal{D}^2 + p_u)}{\rho_u\mathcal{D}^2} \approx -I_2 \end{aligned}}$$

$$I_1 \approx 2\epsilon\int_0^d\left(\frac{\overline{\rho}(x)}{\rho_u} - 1\right)\frac{\mathrm{d}x}{d_{CJ}} \qquad I_2 \approx 2\epsilon\int_0^d\left(1 - \frac{\rho_u}{\overline{\rho}(x)}\right)\frac{\mathrm{d}x}{d_{CJ}}$$

$$\left(\frac{\gamma}{\gamma-1}\frac{p_b}{\rho_b} + \frac{u_b^2}{2}\right) \approx \left(\frac{\gamma}{\gamma-1}\frac{p_u}{\rho_u} + \frac{\mathcal{D}^2}{2} + q_m\right)$$

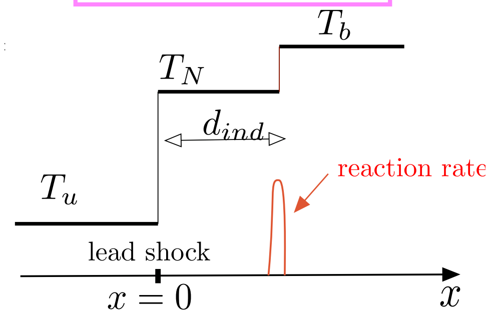
Square-wave model: thickness of the reaction zone $\ll$ thickness of the induction zone $d_{ind}$,

$\boxed{d \approx d_{ind}, \quad I_{1,2} \propto \epsilon\, d_{ind}/d_{CJ}}$

Arrhenius law $\Rightarrow \beta_N \equiv \dfrac{E}{k_BT_{N_{CJ}}} \gg 1,$ $\boxed{d_{ind} = d_{CJ}\exp\left[-2\beta_N\dfrac{(\mathcal{D} - \mathcal{D}_{CJ})}{\mathcal{D}_{CJ}}\right]}$ $\quad \dfrac{(\mathcal{D} - \mathcal{D}_{CJ})}{\mathcal{D}_{CJ}} = O(1/\beta_N)$

$$(\gamma - 1)M_u^2 \gg 1 \Rightarrow \frac{\rho_u}{\rho_N} \approx \frac{\gamma-1}{\gamma+1}, \quad \frac{T_N}{T_u} \approx 2\gamma M_u^2\frac{(\gamma-1)}{(\gamma+1)^2}, \quad \frac{T_N}{T_{N_{CJ}}} \approx \left(\frac{\mathcal{D}}{\mathcal{D}_{CJ}}\right)^2$$

$$\boxed{I_1(\mathcal{D}) \approx \frac{4}{\gamma-1}\frac{d_{CJ}}{r_f}\exp\left[-2\beta_N\frac{(\mathcal{D} - \mathcal{D}_{CJ})}{\mathcal{D}_{CJ}}\right], \qquad I_2 \approx \frac{\gamma-1}{\gamma+1}I_1}$$



## Small modification of the burnt gas state

$$\frac{d_{CJ}}{r_f} = O(1/\beta_N) \qquad \delta\mathcal{D}/\mathcal{D}_{CJ} \equiv (\mathcal{D}-\mathcal{D}_{CJ})/\mathcal{D}_{CJ} = O(1/\beta_N) \quad \Rightarrow \quad I_1 = O(1/\beta_N), \quad I_2 = O(1/\beta_N)$$

$$\Rightarrow \quad \delta u_b(\mathcal{D})/a_{b_{CJ}} = O(1/\beta_N), \qquad \delta\rho_b(\mathcal{D})/\rho_{b_{CJ}} = O(1/\beta_N), \qquad \delta p_b(\mathcal{D})/p_{b_{CJ}} = O(1/\beta_N)$$

$$\delta u_b \equiv u_b - a_{b_{CJ}}, \qquad \delta\rho_b \equiv \rho_b - \rho_{b_{CJ}}, \qquad \delta p_b \equiv p_b - p_{b_{CJ}}$$

Small variations of the continuity eq, the Euler eqs and the energy eq yield

$$\frac{\delta\rho_b}{\rho_{b_{CJ}}} + \frac{\delta u_b}{a_{b_{CJ}}} = \frac{\delta\mathcal{D}}{\mathcal{D}_{CJ}} - I_1 \qquad \frac{1}{\gamma}\frac{\delta p_b}{p_{b_{CJ}}} + \frac{\delta\rho_b}{\rho_{b_{CJ}}} + 2\frac{\delta u_b}{a_{b_{CJ}}} = \left(\frac{\mathcal{D}_{CJ}}{a_{b_{CJ}}}\right)\left(2\frac{\delta\mathcal{D}}{\mathcal{D}_{CJ}} - I_2\right) \qquad \frac{1}{\gamma-1}\frac{\delta p_b}{p_{b_{CJ}}} - \frac{1}{\gamma-1}\frac{\delta\rho_b}{\rho_{\rho_{CJ}}} + \frac{\delta u_b}{a_{b_{CJ}}} = \left(\frac{\mathcal{D}_{CJ}}{a_{b_{CJ}}}\right)^2\frac{\delta\mathcal{D}}{\mathcal{D}_{CJ}}$$

Sonic condition in the burnt gas : $\quad u_b^2 = \gamma p_b/\rho_b \quad \Rightarrow \quad \dfrac{\delta p_b}{p_{b_{CJ}}} - \dfrac{\delta\rho_b}{\rho_{\rho_{CJ}}} - 2\dfrac{\delta u_b}{a_{b_{CJ}}} = 0$

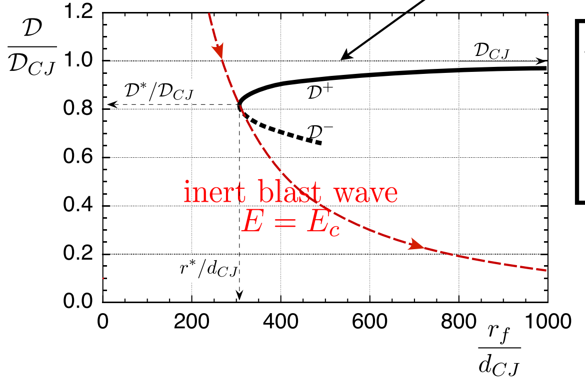
$$\frac{(\rho_b u_b - \rho_u\mathcal{D})}{\rho_u\mathcal{D}} \approx -I_1$$
$$\frac{(\rho_b u_b^2 + p_b) - (\rho_u\mathcal{D}^2 + p_u)}{\rho_u\mathcal{D}^2} \approx -I_2$$
$$\left(\frac{\gamma}{\gamma-1}\frac{p_b}{\rho_b} + \frac{u_b^2}{2}\right) \approx \left(\frac{\gamma}{\gamma-1}\frac{p_u}{\rho_u} + \frac{\mathcal{D}^2}{2} + q_m\right)$$

$$\left\{\frac{\gamma+1}{\gamma}\left[1 + \frac{\gamma-1}{\gamma+1}\left(\frac{\mathcal{D}_{CJ}}{a_{b_{CJ}}}\right)^2\right] - 2\frac{\mathcal{D}_{CJ}}{a_{b_{CJ}}}\right\}\frac{\delta\mathcal{D}}{\mathcal{D}_{CJ}} = \frac{\gamma+1}{\gamma}I_1(\mathcal{D}) - \frac{\mathcal{D}_{CJ}}{a_{b_{CJ}}}I_2(\mathcal{D})$$

$$\frac{\mathcal{D}_{CJ}}{a_{CJ}} \approx \frac{\gamma+1}{\gamma}$$

$$I_1(\mathcal{D}) \approx \frac{4}{\gamma-1}\frac{d_{CJ}}{r_f}\exp\left[-2\beta_N\frac{(\mathcal{D}-\mathcal{D}_{CJ})}{\mathcal{D}_{CJ}}\right], \qquad I_2 \approx \frac{\gamma-1}{\gamma+1}I_1 \quad \Rightarrow \quad \boxed{2\beta_N\left(\frac{\mathcal{D}_{CJ}-\mathcal{D}}{\mathcal{D}_{CJ}}\right)\mathrm{e}^{-2\beta_N\left(\frac{\mathcal{D}_{CJ}-\mathcal{D}}{\mathcal{D}_{CJ}}\right)} = \frac{16\gamma^2}{\gamma^2-1}\beta_N\frac{d_{CJ}}{r_f}}$$

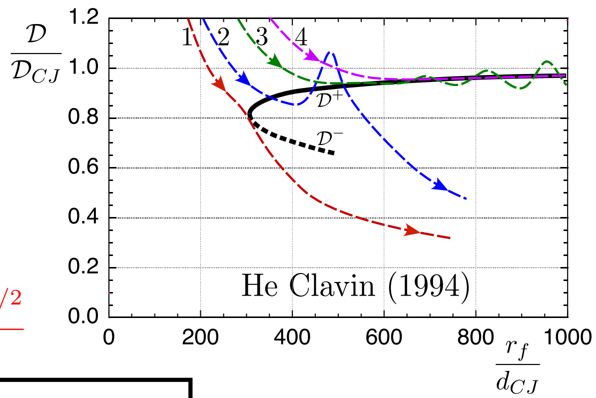
$\left(\dfrac{\mathcal{D}_{CJ}-\mathcal{D}}{\mathcal{D}_{CJ}}\right)$ vs $\dfrac{d_{CJ}}{r_f}$ is a C-shaped curve with a turning point at $\boxed{r^* = \dfrac{16\,\mathrm{e}\,\gamma^2}{\gamma^2-1}\beta_N\, d_{CJ}, \qquad \mathcal{D}^* = \left(1 - \dfrac{1}{2\beta_N}\right)\mathcal{D}_{CJ}}$




> there is no spherical CJ detonation with a radius $r_f < r^*$
>
> $r_f^*/d_{CJ} \approx 10^3$

inert blast wave $\quad D \propto \dfrac{(E/\rho_u)^{1/2}}{r^{3/2}}$

marginal blast wave $\quad D_{CJ} \propto \dfrac{(E_c/\rho_u)^{1/2}}{r^{*3/2}}$



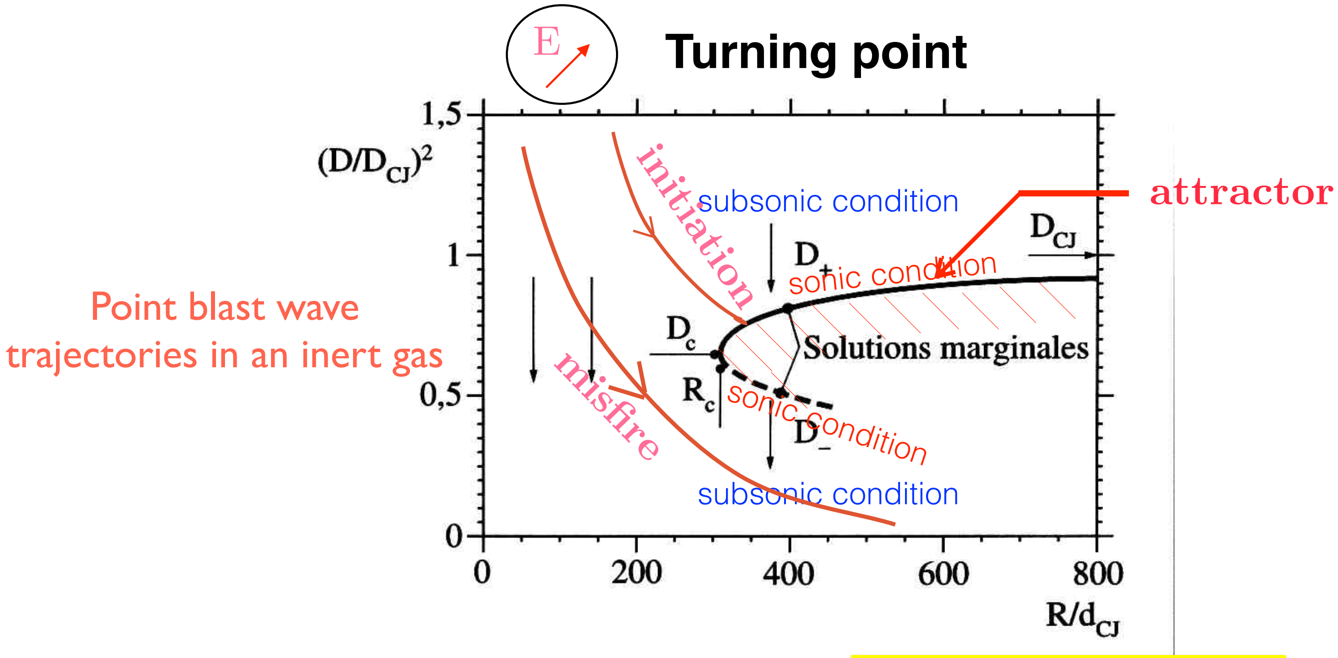

$$\boxed{Ec \approx (5/2)^2\rho_u\mathcal{D}_{CJ}^2 r^{*3} = (5^2/2)(\gamma^2-1)q_m\rho_u r^{*3} \approx 10^8 \text{ - } 10^9 \times \text{Zeldovich value (1956)}}$$

ok with DNS of He Clavin (1994)
ok with the experiments of Lee (1984)

**Limitations of the analysis:** Square-wave model. Quasi-steady state

**No change in order of magnitude**

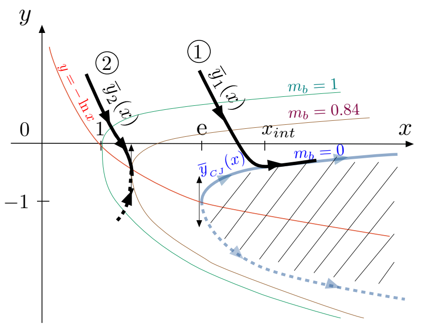
## Curvature effect on the detonation structure
## (**steady state** approximation)




$$\frac{\text{critical radius}}{\text{detonation thickness}} \approx 10^2 \quad \Rightarrow \quad \boxed{\frac{\text{critical energy}}{\text{Zeldovich value}} \approx 10^6}$$

Arrhenius factor

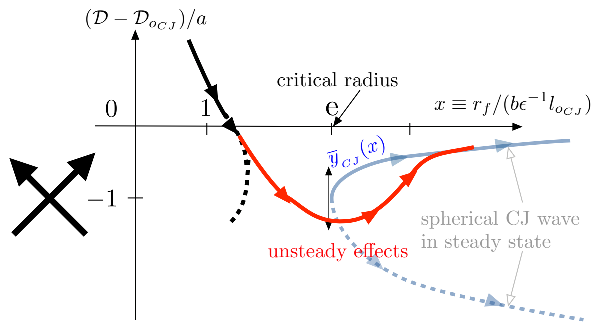
Good order of magnitude but the critical energy is overestimated !




# Weakness of the analysis:

quasi-steady state approximation

*He Clavin JFM (1994)* ***277*** *p. 227-248*



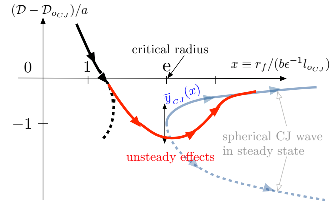

unsteadiness induced re-ignition for $R < R_c$

# Lecture 11-a: Direct initiation

*Critical dynamics*

**Unsteady analysis** of the direct initiation in a spherical geometry
in the limit of **small heat release** enlightening the **qualitative** behavior

$$\epsilon \equiv q/c_p T_u < 1$$




Clavin, Denet *JFM* (2020) Vol. **897**, *A*30

Clavin, Hernandez-Sanchez and Denet *JFM (*2021) Vol. **915,** A 122

**1st step**: Asymptotic analysis of the **rarefaction wave** in the **discontinuous** model
(**detonation = discontinuity**)

- ***Self-similar*** *solution behind a spherical CJ wave*
- ***Unsteady rarefaction wave*** *behind overdriven detonations approaching CJ*
- ***Transient flow*** *when reaching the CJ velocity*

**2nd step**: **Unsteady inner structure** of the detonation taken into account

- ***Two-length scales*** *problem: matching condition*
- ***Critical dynamics:*** *the role of the trajectory of the sonic point*

**1st step**: Asymptotic analysis of the rarefaction wave in the discontinuous model
(detonation = discontinuity)

- ***Self-similar*** *solution for the reaction wave behind a spherical CJ wave*

asymptotic analysis in the limit of **small heat release** $\epsilon \equiv q/c_p T_u < 1$ **M-1 << 1**

Clavin, Hernandez and Denet, *JFM* (2021) Vol. **915**, *A*122

$$0 < (\mathcal{D}_{o_{CJ}} - a)/a \approx \epsilon \ll 1, \qquad \mathcal{D}_{o_{CJ}} \approx (1+\epsilon)a,$$

the result is qualitatively similar to the limit of large Mach number M >> 1 by Zeldovich (1942) Taylor (1945)

detonation front: $r = r_f(t) > r_0(t) \qquad \mathrm{d}r_f/\mathrm{d}t = \mathcal{D}_{o_{CJ}} > a_o$

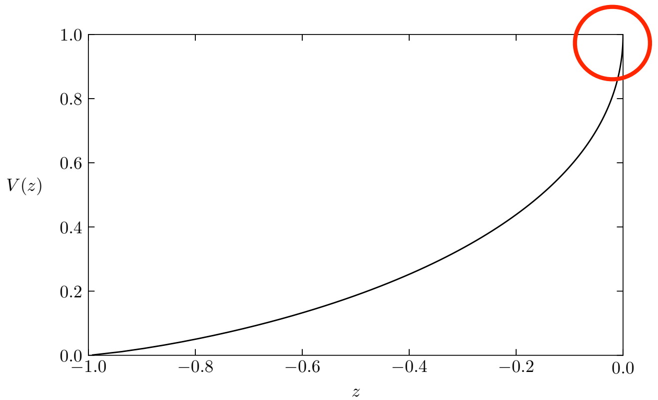
inert core

$$r_0 \leqslant r \leqslant r_f(t) : \quad \frac{u}{\epsilon a} = V\left(\frac{r - r_f(t))}{\epsilon r_0(t)}\right)$$

analytical solution of the flow field

$$z \equiv \frac{r - r_f(t))}{\epsilon r_0(t)}$$

$$\boxed{V(z)\ln V(z) - V(z) + (z+1) = 0}$$




inert core radius

CJ front

**Same singularity** on the **detonation front** as in Zeldovich Taylor !

# 1st step: Asymptotic analysis of the rarefaction wave in the discontinuous model

limit of ***small heat release***

Clavin, Hernandez and Denet, *JFM* (2021) Vol. **915**, *A*122

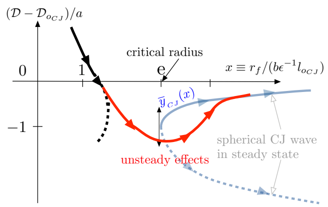
## - Unsteady rarefaction wave behind *overdriven detonations approaching CJ*

radius of the inner core of burned gas at rest

flow velocity of burned gas at the detonation front

$$0 < \frac{u_{fi}}{\epsilon a} - 1 \ll 1, \quad r_0(t) \leqslant r \leqslant r_f(t): \quad \frac{u(r,t)}{u_f(t)} = \frac{r - r_0(t)}{r_f(t) - r_0(t)} \qquad r_0(t) = a_o t + r_{0i}, \qquad u_f(t) \equiv u(r_f, t)$$

detonation front




$$u_f(t) \geqslant 1: \quad \frac{u_f(t)}{u_{fi}} = \frac{r_f(t) - r_0(t)}{r_{fi} + a_o t}\left[\frac{r_{fi} - r_{0i}}{r_{fi}} + \frac{u_{fi}}{a_o}\ln\left(1 + \frac{a_o t}{r_{fi}}\right)\right]^{-1}$$

CJ value

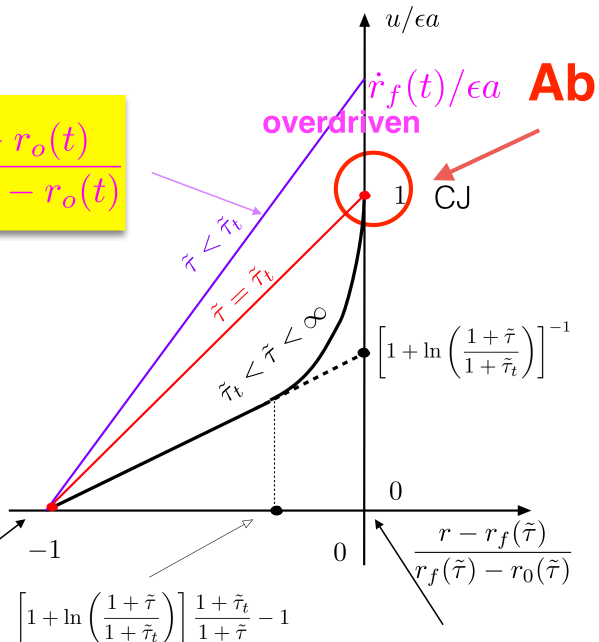
**linear velocity profile** (strictly limited to overdriven regimes)

## - Transient flow of the rarefaction wave as soon as the CJ velocity is reached

Overdriven solution ⟶ Self similar CJ solution

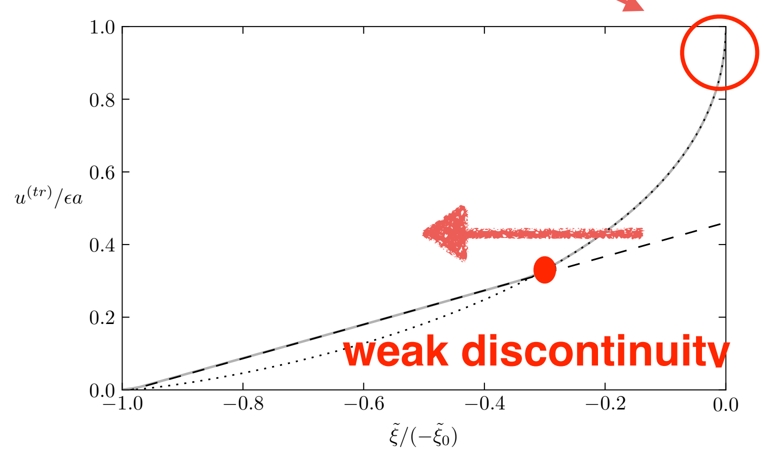
**Abrupt transition** of the gradient on the detonation fror

$$\frac{u(r,t)}{u_f(t)} = \frac{r - r_o(t)}{r_f(t) - r_o(t)}$$




radius of the core of burned gas at rest

detonation front



## 2nd step: **Unsteady inner structure** of the detonation taken into account

**Analytical solution** in the limit of **small heat release** $\epsilon \equiv q/c_p T_u < 1$

Clavin, Hernandez and Denet, *JFM* (2021) Vol. **915**, *A*122

One-step model+ Arrhenius law+temperature cutoff

**Two length scales problem.** Flame thickness = small scale. Rarefaction wave = long scale

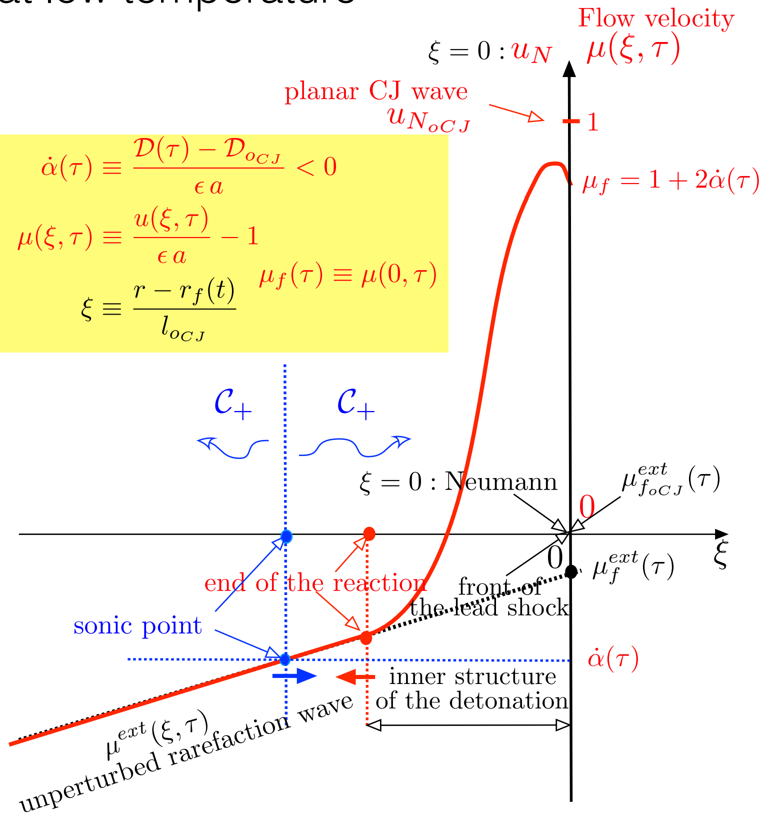
(internal flame structure) (external flow)

In the limit of small heat release, the problem is reduced to a **single nonlinear** differential equation of first order for the flow field that has to be solved by **matching** the solution in the burned gas side of the inner flame structure with a **point blast wave**. The **Arrhenius law** governing the reaction rate is truncated below a **crossover temperature** denoting the chemical kinetics induced **quenching** at low temperature

propagation velocity of the lead shock $\dot\alpha(\tau) \equiv \dfrac{\mathcal{D}(\tau) - \mathcal{D}_{oCJ}}{\epsilon\, a} < 0$

flow field $\mu(\xi,\tau) \equiv \dfrac{u(\xi,\tau)}{\epsilon\, a} - 1$

$|\xi| =$ distance from the lead shock $\xi \equiv \dfrac{r - r_f(t)}{l_{oCJ}}$ $\mu_f(\tau) \equiv \mu(0,\tau)$

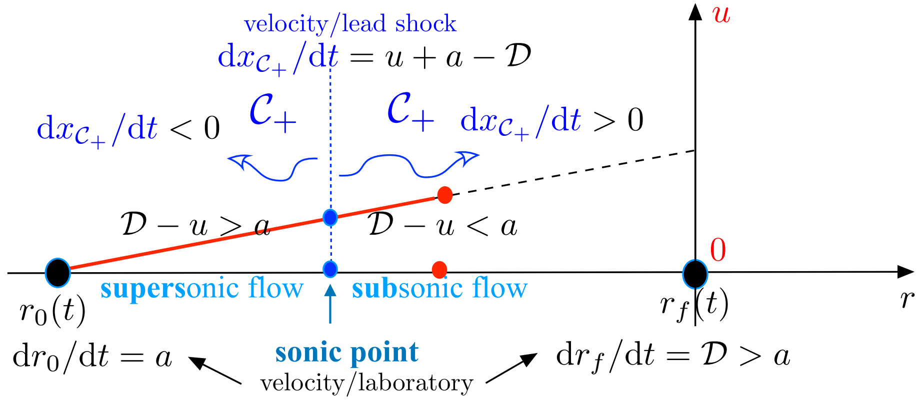


## **Critical dynamics:** Role of the trajectory of the sonic point

Propagation velocity (**relative to the lead shock**)
**of the disturbances associated with** $\mathcal{C}_+$ is

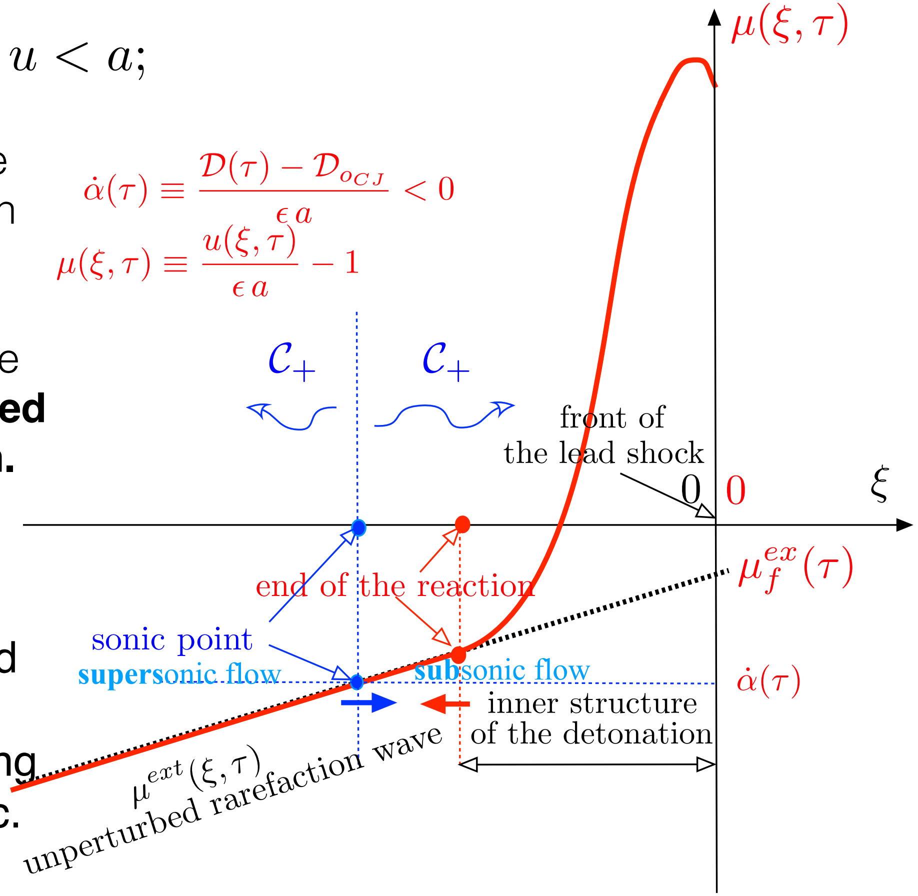
$$\mathrm{d}x_{\mathcal{C}_+}/\mathrm{d}t = u + a - \mathcal{D}$$

$$\text{sonic point: } \mathcal{D} - u = a$$




The **sonic point** separates the flow in two regions:
**ahead**, the flow is **subsonic** (relative to the lead shock), $\mathcal{D} - u < a$;
**behind**, the flow is **supersonic**, $\mathcal{D} - u > a$.

The **rarefaction** wave is **not disturbed** behind the exit of the reaction zone by the heat-release: it is the same as behind an overdriven detonation of zero thickness for which the flow gradient is uniform and decreases continuously. The flow of burned gas adjacent to the reaction zone being **subsonic** the detonation regime is overdriven. The **reaction rate is decreased** by the rarefaction wave and the **detonation is slowed down.**

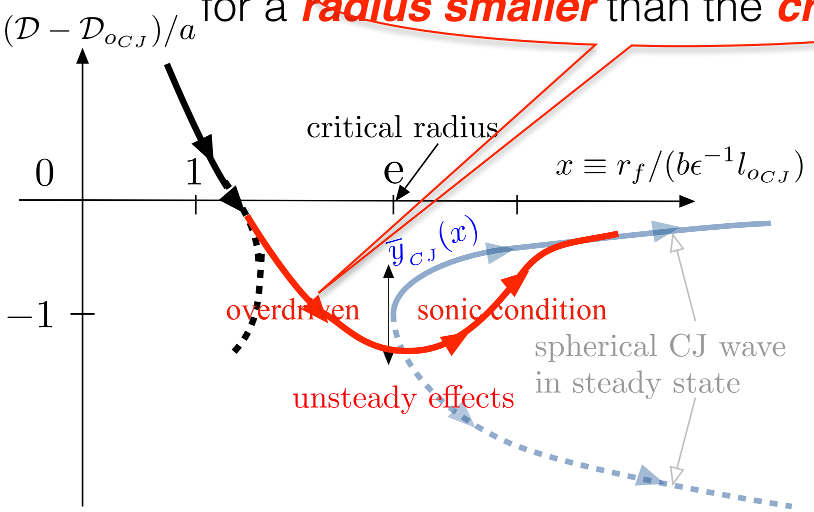


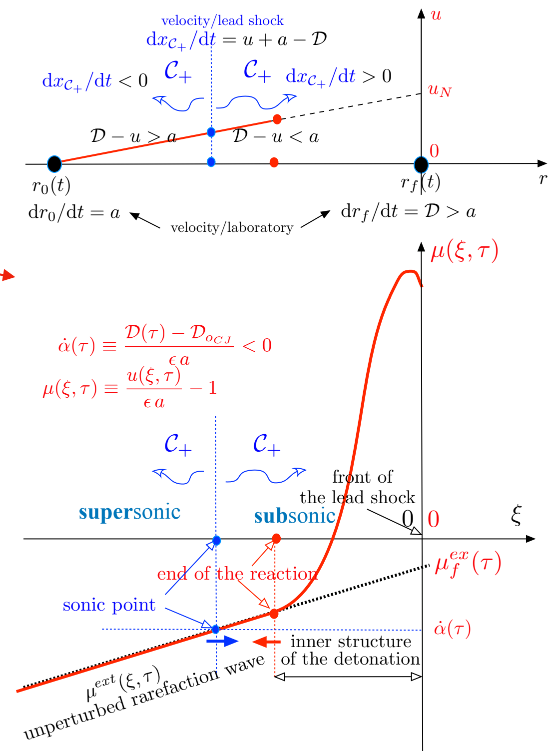

The flow gradient of the rarefaction wave decreasing continuously, the sonic point **get closer and closer to the exit of the reaction zone**. The damping is stopped as soon as the sonic point catches the exit of the reaction zone protecting the reaction rate of further damping since the flow relative to the reaction zone becomes sonic. This is possible if the gas temperature in the rarefaction wave has not decreased below the cross over temperature

When the sonic point **enters** the **reaction zone**, it **stays stuck near the end of the reaction** and the sonic condition makes the detonation **free from further damping** by the rarefaction wave. The inner structure of the detonation (initially slaved by the external flow) becomes **isolated in a state out of equilibrium**. The equilibrium state is restored through an increase in reaction rate, producing a **re-acceleration of the detonation front** towards the **spherical CJ** regime. This re-acceleration occurs **near the critical radius** characterizing the quasi-steady spherical detonations.

Therefore, the **critical energy is overestimated** (of an order unity) by the nonlinear curvature effect of the spherical CJ detonation in quasi-steady state **but the critical radius** is not modified

Sketch of the **flow field** of an **overdriven detonation** for a velocity of the lead shock ***below*** the planar CJ value, for a ***radius smaller*** than the ***critical radius***



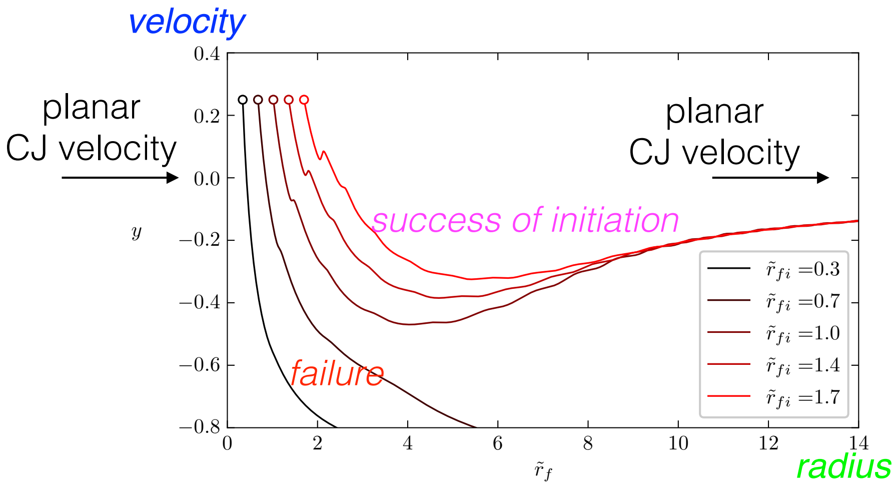



2nd step: Unsteady inner structure of the detonation taken into account

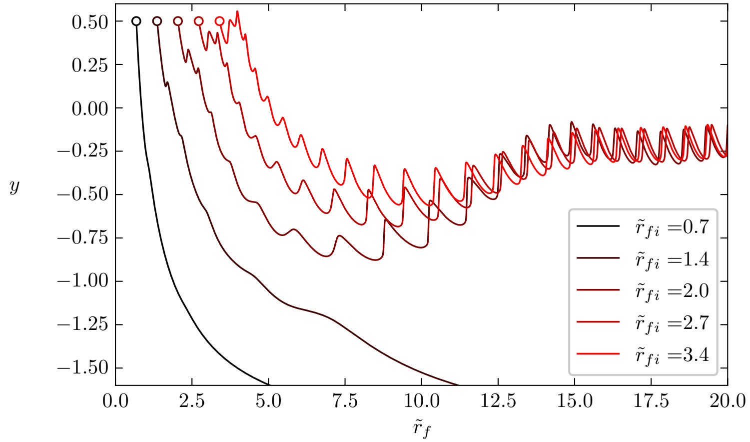
Analytic solution in the limit of **small heat release** $\epsilon \equiv q/c_pT_u < 1$

Clavin, Hernandez and Denet, *JFM* (2021) Vol. **915**, *A*122

The trajectories « *velocity of the lead shock versus radius* » are obtained near the critical condition by the numerical study of the single equation of the asymptotic analysis. Typical results are plotted below for different initial conditions corresponding to blast waves with different deposited energy.



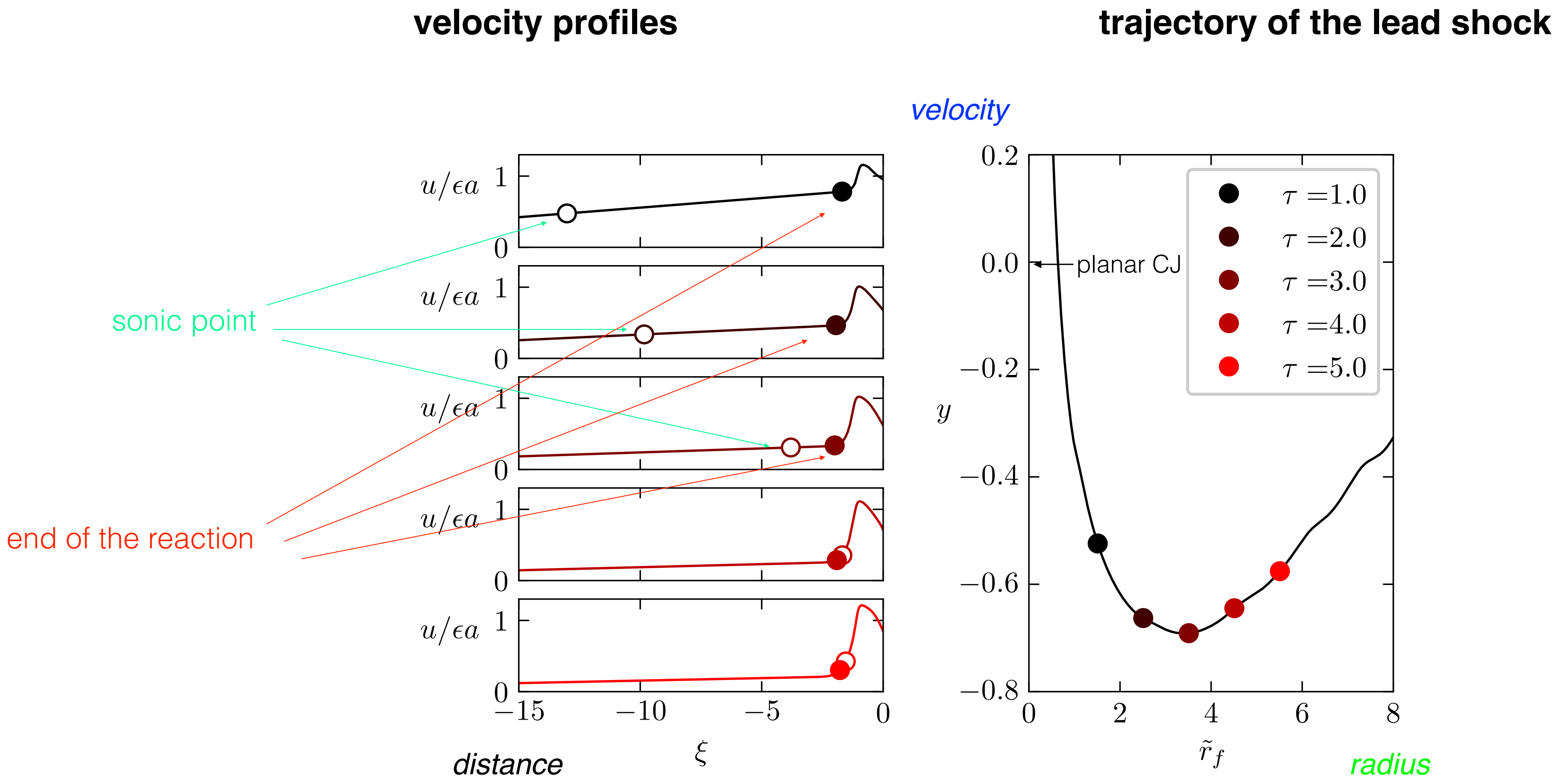

*stable detonation*




*unstable detonation*

Success of initiation occurs if the sonic point reaches the reaction zone before the reaction gets thermally quenched. Otherwise failure is produced

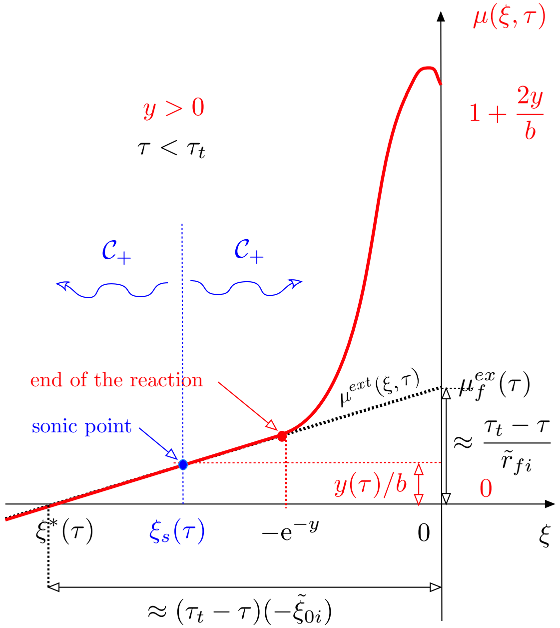
# Numerical solution of the equation




Damping by the rarefaction wave **stops** as soon as the **sonic point enters the reaction zone** and protects the detonation structure from further damping by the burnt gas flow, then the front **re-accelerates**

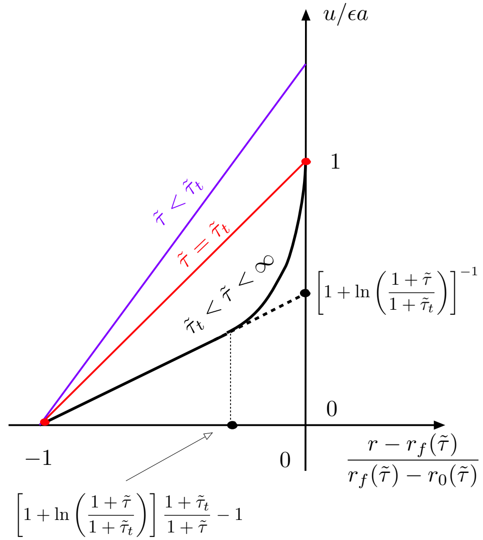
## Analytical solution with the modification of the inner detonation structure

Curvature + unsteadiness in the limit of small heat release




Full-solution
Unsteady inner detonation structure

$\times$

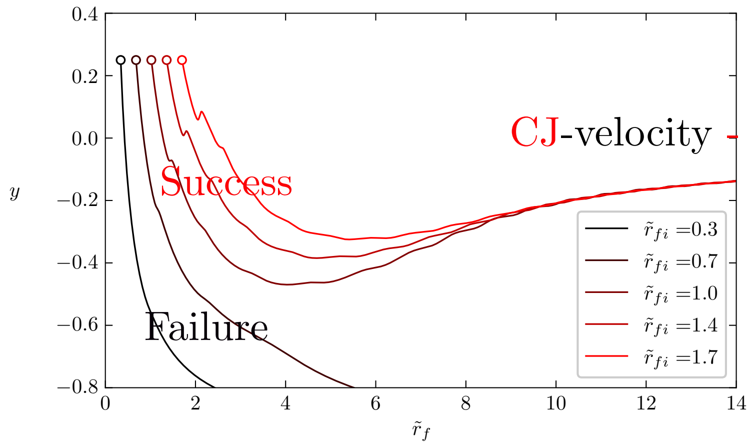



Rarefaction wave: zero-thickness-detonation
Unsteady-solution

# Analytical solution with the modification of the inner detonation structure

Curvature + unsteadiness in the limit of small heat release

## Critical dynamics between failure and initiation




Trajectories velocity-radius

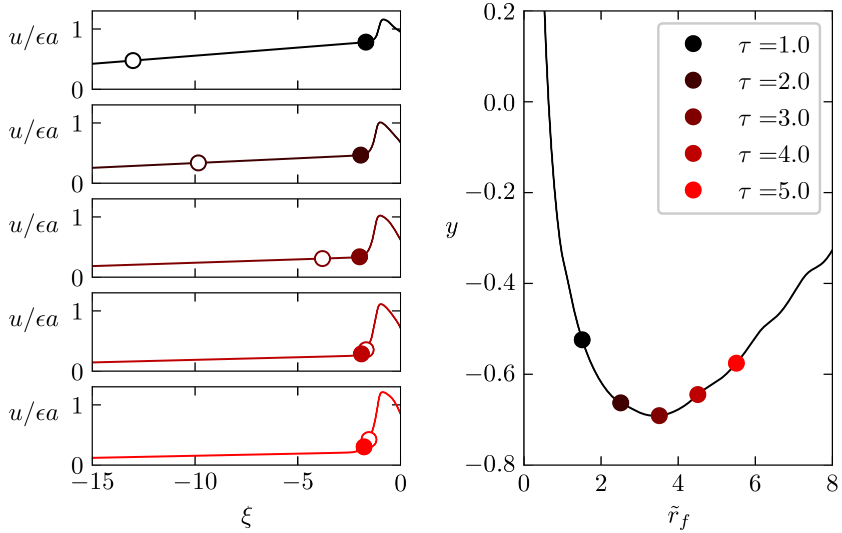
Re-ignition after a quasi-quenching

## Overdriven regime below the planar CJ velocity

relative position

sonic point

exit of the reaction zone




sonic point vs exit of the reaction zone

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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Aix-Marseille Université
**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

# Lecture XI
# Initiation of detonations
# Part 2

# Lectures 11: Initiation of detonation

## Lecture 11-a: Direct initiation

*Flow of burnt gas in spherical CJ detonations*
*Point blast explosions*
*Critical energy*
*Critical dynamics*

## Lecture 11-b: Spontaneous initiation and quenching

*Initiation at high temperature*
*Spontaneous quenching*

## Lecture 11-c: Deflagration to Detonation Transition

*Basic ingredients*
*Experiments*
*Runaway phenomenon*
*Intrinsic DDT mechanism of laminar flames*

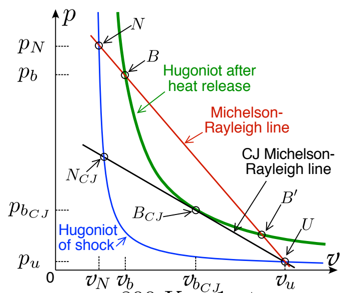
# Spontaneous initiation of detonation at high temperature

Zeldovich (1970-1980) Lee (1978)

Ya. B. Zel'dovich et al. (1970) *J. Appl. Mech. & Tech. Phys.* **11**, pp. 264-270 J.H.S. Lee et al. (1978) *Acta Astronautica* **5**, pp. 971-982

Gaseous detonations are difficult to ignite: a large increase of pressure is required $p_{N_{CJ}} \approx 30 - 50\text{atm}$

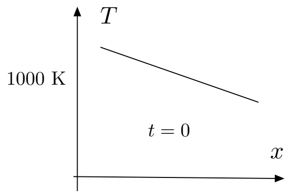
Not possible with an homogeneous explosion of a gaseous pocket at constant volume $(\Delta p/p < 10)$ $(T_N < T_c)$



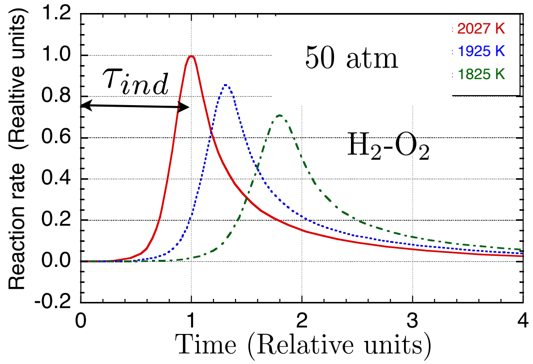

Possible with gradients of $T$ Zeldovich (1970-1980) Lee (1978)

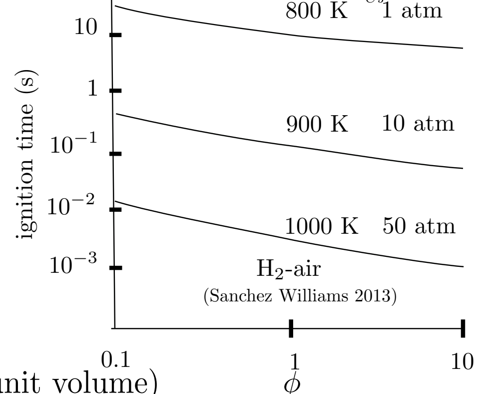
Induction delay (Ignition time) $\tau_{ind}(T,p)$ is highly sensitive to $T$ $\boxed{\tau_{ind} \searrow \quad T \nearrow}$

$t=0$ : initial gradient of $T$ $\Rightarrow$ gradient of $\tau_{ind}$




$\boxed{\tau_{ind}(x)}$




1-$D$ : hot slides ignite before cold slides $\dot{q}_v = q_v \varpi\left(t - \tau_{ind}(x)\right)$ (rate of heat release per unit volume)

dimension of $\varpi$ = (reaction time)$^{-1}$

$\Rightarrow$ propagation of an induction front at a speed $\approx (\mathrm{d}\tau_{ind}/\mathrm{d}x)^{-1}$

Mechanism spontaneous initiation: combustion $\Rightarrow$ pressure pulses that propagate with about the speed of sound $a$

synchronisation : $\boxed{(\mathrm{d}\tau_{ind}/\mathrm{d}x)^{-1} = a}$

$\mathrm{d}\tau_{ind}/\mathrm{d}x \approx \text{cst.} \quad \Rightarrow \tau_{ind}(x) \approx \tau^o_{ind} + x(\mathrm{d}\tau_{ind}/\mathrm{d}x)$ $\quad \dot{q}_v = q_v\varpi\left(t - \tau^o_{ind} - x(\mathrm{d}\tau_{ind}/\mathrm{d}x)\right) \approx q_v\varpi\left(t - \tau^o_{ind} - x/a\right)$

$\partial^2 p/\partial t^2 - a^2\partial^2 p/\partial x^2 = (\gamma-1)\partial\dot{q}_v/\partial t$ simple wave : $\dfrac{\partial}{\partial t}\delta p + a\dfrac{\partial}{\partial x}\delta p = (\gamma-1)q_v\varpi\left(t - \tau^o_{ind} - x/a\right)$

run away (secular solution) $\delta p = t(\gamma-1)q_v\varpi\left(t - \tau^o_{ind} - x/a\right)$

**The amplitude of the pressure pulse increases linearly with time at the rate of the reaction rate**

$$\boxed{(\mathrm{d}\tau_{ind}/\mathrm{d}x)^{-1} = a}$$

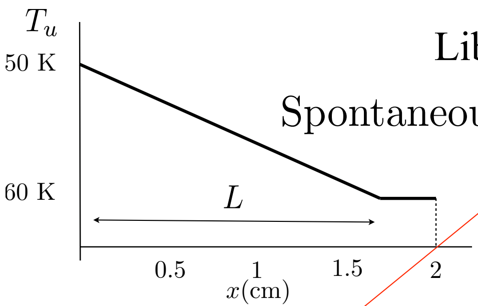
Spontaneous initiation has been observed in experiments and numerics

Asymptotic analysis: Sharpe & Short (2003) *JFM*, **476**, 267-292

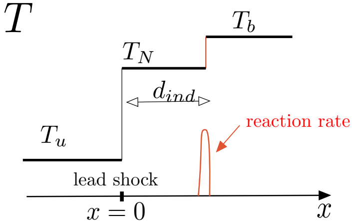
# Spontaneous quenching

He, Clavin *(1994) Proc.Comb.Inst.* **25** *p. 45-51*

Liberman et al. *(2018) Combust. Theory Model.* **23** *(3)p. 4183-4193*

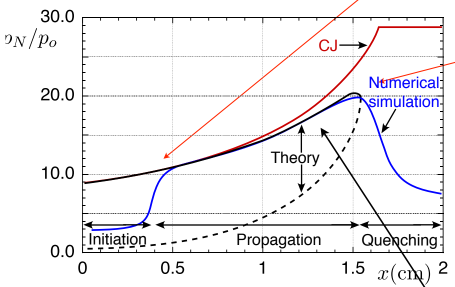


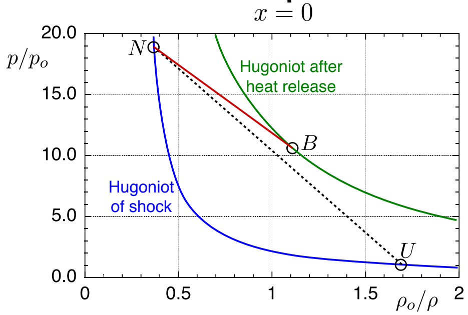

Spontaneous ignition at high $T$ may be followed by sudden quenching at lower $T$




Theoretical analysis:

- square-wave model
- quasi-steady induction zone : $d_{ind}/\overline{d}_{ind} \approx \mathrm{e}^{-2\beta_N \frac{(\mathcal{D}-\overline{\mathcal{D}})}{\overline{\mathcal{D}}}}$
- $\mathrm{d}d_{ind}/\mathrm{d}t \neq 0 \Rightarrow$ mass flux in reaction zone $\neq \rho_u \mathcal{D}$



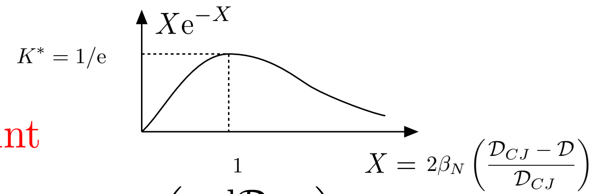




$$\frac{\delta D_{CJ}}{D_{CJ}} \approx (2M_{u_{CJ}}^2)^{-1} \frac{\delta T_u}{T_u} \Rightarrow \boxed{\frac{\mathrm{d}}{\mathrm{d}t} d_{ind} \propto \mathrm{e}^{-2\beta_N \frac{(\mathcal{D}-\overline{\mathcal{D}})}{\overline{\mathcal{D}}}} L \frac{\mathrm{d}\mathcal{D}}{\mathrm{d}x}}$$

geometrical construction:

difference of mass flux = difference of slopes of $UN$ and $NB \propto (\mathcal{D}-\overline{\mathcal{D}})/\overline{\mathcal{D}}$ $\quad$ ( $(\gamma-1)M_u^2 \gg 1$ )

$$(\gamma-1)M_u^2 \gg 1 : \quad \boxed{\frac{\mathrm{d}}{\mathrm{d}t} d_{ind} \propto \overline{u}_N \frac{(\mathcal{D}-\overline{\mathcal{D}})}{\overline{\mathcal{D}}}}$$




C-shaped curve $\mathcal{D}$ vs $\mathrm{d}T_u/\mathrm{d}x$ with a turning point

$$\boxed{2\beta_N \left(\frac{\mathcal{D}_{CJ}-\mathcal{D}}{\mathcal{D}_{CJ}}\right) \mathrm{e}^{-2\beta_N \left(\frac{\mathcal{D}_{CJ}-\mathcal{D}}{\mathcal{D}_{CJ}}\right)} = K}$$

where $\quad K \equiv 4\beta_N^2 \tau_{ind}(T_{N_{CJ}}) \left(-\frac{\mathrm{d}\mathcal{D}_{CJ}}{\mathrm{d}x}\right) = O(1)$

critical condition for sudden quenching $\quad K^* = 1/\mathrm{e} \quad \left(\frac{\mathcal{D}_{CJ}-\mathcal{D}^*}{\mathcal{D}_{CJ}}\right) = \frac{1}{2\beta_N} \quad$ ok with DNS

A CJ detonation cannot survive to a strong temperature gradient at low temperature $\quad (K > 1/\mathrm{e})$ $\quad \frac{\mathrm{d}K}{\mathrm{d}T_u} < 0$

The non-uniform pocket of hot gas should have a proper shape for initiating a detonation

# Lectures 11: Initiation of detonation

## Lecture 11-a: Direct initiation

*Flow of burnt gas in spherical CJ detonations*
*Point blast explosions*
*Critical energy*
*Critical dynamics*

## Lecture 11-b: Spontaneous initiation and quenching

*Initiation at high temperature*
*Spontaneous quenching*

## Lecture 11-c: Deflagration to Detonation Transition

*Basic ingredients*
*Shchelkin scenario*
*Thermal feedback*
*Recent experiments*
*Intrinsic DDT mechanism of laminar flames*
*(Runaway of the flame structure)*

# Lecture 11-c: Deflagration to Detonation Transition (DDT)

*Basic ingredients*

DDT is an abrupt transition between two opposite regimes of propagation

Flame = reaction-diffusion wave markedly subsonic, $U_L/a_u \ll 1$

Detonation = shock driven supersonic combustion wave without diffusion behind the lead shock, $\mathcal{D}/a_u > 1$

No intermediate quasi steady propagation regime !

Despite more than a century of researches, DDT remains a poorly understood problem

Since the pioneering experiments of Oppenheim (1965), DDT has been known to develop in various forms (viscous effects in the boundary layers, unsteady compression waves, thermal gradients, local explosion…)

There is no mechanism of DDT that is generally agreed upon as being universal !

However it is now well established that DDT is a local and sudden explosion of a small hot spot either on the flame front or ahead of it, either inside or outside the boundary layer on the wall

(Urtiew and Oppenheim (1966) *Proc. R. Soc. London* **A 295** pp 13-28)

Here the attention will be limited to an intrinsic DDT mechanism of laminar flames:

DDT of elongated laminar flames propagating in closed tubes on the burned gas side

Turbulence can promote an early transition but is not an essential DDT mechanism

## First, few words about DDT induced by local explosions in the boundary layer

(Oppenheim's experiments 1966-1973)

Urtiew & Oppenheim (1966) *Proc. R. Soc. London* **A 259**, pp. 13-28

Hot spots in the boundary layer between the lead shock and the flame

Heating by viscous dissipation in the boundary layers at the wall

+

compressional heating

$\nabla T \neq 0$ $\qquad\qquad$ $\nabla$ induction time $\neq 0$

**local explosion ahead of the flame by the spontaneous ignition of Zeldovich**

(we will come back to this case at the end of the lecture)

---

Here the attention will be limited to the DDT near the tube axis on the leading edge of an elongated front of a laminar flame propagating in a tube

**Experiments in** $50 \times 50\ \mathrm{mm}^2$ **closed** channels:

Liberman et al. (2010) *Acta Astronautica* **67**, 688-701

Kuznetsov et al. (2010) *Combust Sci. Tech.*, **182**, 1628-1644

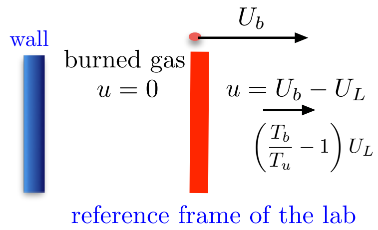
# Basic ingredients of the intrinsic DDT mechanism of laminar flames ignited on the closed wall of a tube

## Burned gas at rest

flame speed = laminar flame/burned gas

$$U_b = (T_b/T_u)U_L$$

## Planar flame




-1) Piston effect: fresh gas put in motion ahead of the flame

$$u = U_b - U_L = \left(\frac{T_b}{T_u} - 1\right)U_L$$

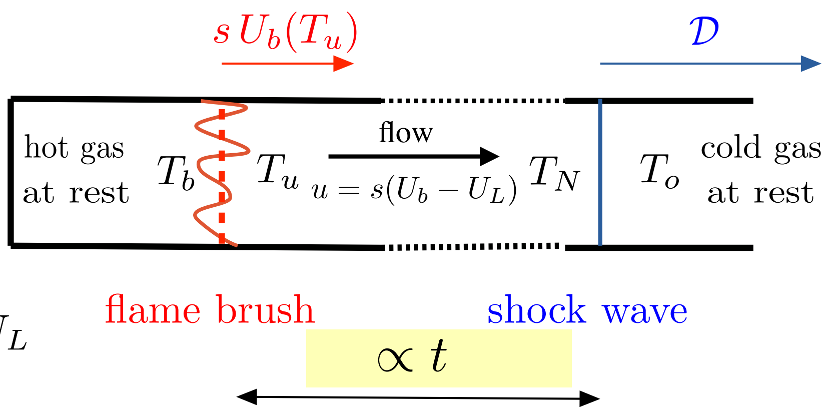
-2) Flame acceleration through an increase of the flame surface area

Turbulent wrinkled flame

folding factor: $\quad s = \dfrac{\text{flame surface area}}{\text{cross section}}$

$$U_b \to sU_b \qquad U_L \to sU_L$$

velocity of the unburned gas flow $\quad u = s(U_b - U_L) = \left(\frac{T_b}{T_u} - 1\right)sU_L$



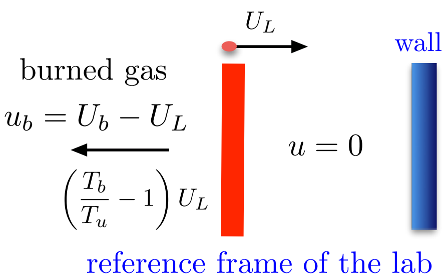

-3) Heating of the fresh mixture by compressible effects
(through a shock wave or a compression wave and/or viscous dissipation)

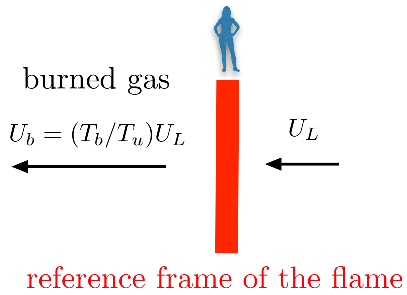
-4) High sensitivity of the flame velocity to the temperature

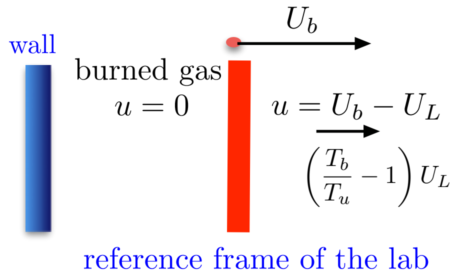
$$U_L(T_u) \qquad \beta \equiv \frac{T_u}{U_L}\frac{\mathrm{d}U_L}{\mathrm{d}T_u} \gg 1 \qquad\qquad T_b = T_u + q_m/c_p \qquad\qquad U_b(T_b) \qquad \frac{T_b}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}T_b} \gg 1$$

# 1-D flame in a tube

**Piston effect** $\dfrac{\rho_u}{\rho_b} = \dfrac{T_b}{T_u} = \dfrac{U_b}{U_u} > 1$

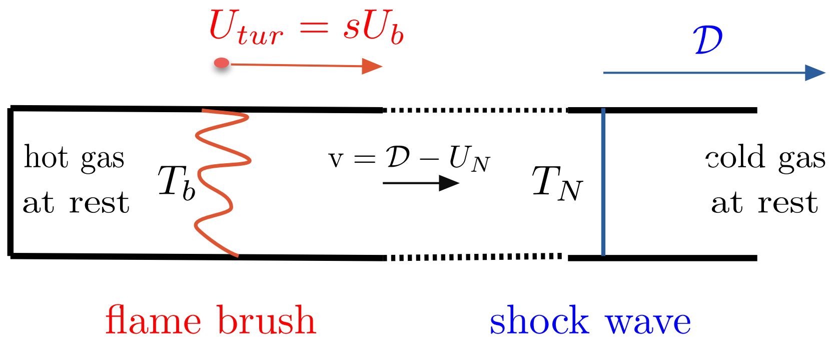



## Turbulent wrinkled flame $U_L \rightarrow s\,U_L$

folding factor: $s = \dfrac{\text{flame surface area}}{\text{cross section}}$




## Turbulence-induced DDT

flow velocity

$s(U_b - U_L)$

the flow upstream of the flame becomes turbulent

the wrinkling of the flame front increases $s \nearrow$

## Runaway mechanism

$s \nearrow$ $\text{v} \nearrow$ turbulence $\nearrow$ wrinkling s $\nearrow$

$\text{v} \nearrow$ $M \equiv \mathcal{D}/a_u \nearrow$ $T_N \nearrow$

M $\nearrow$ 6 $T_N \nearrow$ $1200K$

ignition behind the choc $\Rightarrow$ detonation initiation

**Shchelkin scenario (1940-1950)**

# Lecture 11-c: **Deflagration to Detonation Transition**

## *Shchelkin scenario*

*Turbulence induced DDT* (Shchelkin 1940-1945)

turbulent flow of fresh mixture $\Rightarrow s\nearrow \Rightarrow U_{tur}\nearrow \Rightarrow$ flow velocity$\nearrow \Rightarrow$ turbulence intensity$\nearrow \Rightarrow s\nearrow \Rightarrow U_{tur}\nearrow$

(positive feedback)

~~Strong shock wave $M \geqslant 5 \quad \Rightarrow \quad$ short induction time~~

~~Ignition of the compressed gas~~

---

Not observed in the DDT experiments of elongated flames in tubes $> 1960$

Sudden DDT for $M \approx 3$ temperature of the gas in the fresh mixture too small

fast self-ignition is not possible $(T_N < T_c)$

Sudden DDT not explained

Recent advances

The turbulence-induced DDT scenario is not observed in the 2010-2011 experiments

Millimiter-scale tubes, Wu et al. (2007-2011)

Centimeter-scale tubes, Liberman et al. (2010-2011)

(experiments and and DNSs)

Very energetic stoichiometric $H_2 - O_2$ mixtures: $U_L(T_u) \approx 9$ m/s , $T_b \approx 3000$K ,

$\mathcal{D}_{CJ} \approx 2800$ m/s , $T_{b_{CJ}} \approx 3600$ K

Sudden transition at the flame front in the laminar regime

with a jump of the velocity of the combustion front from $\approx 300$ m/s

to $\approx 3000$ m/s during $10^{-6}$ s.

$\cancel{\phantom{x}}$ $10^{-3}$ s.

at a temperature of the fresh mixture not high enough $T \approx 650$ K

and a Mach number of the lead shock not large enough $M \approx 2.4$

for self-ignition behind the lead shock

This sudden transition was left unexplained. This is the topic of the rest of this lecture

# Lecture 11-c: Deflagration to Detonation
## *Thermal feedback*

## Basic mechanism: Nonlinear thermal loop

Deshaies and Joulin (1989) *Combust. Flame*, **77**, 201-212

*enlightening theoretical analysis ignored up to 2017*

Self-similar solution (1-D) of the double discontinuity model for a turbulent flame in the wrinkled flame regime considered as a planar wave propagating from a closed wall

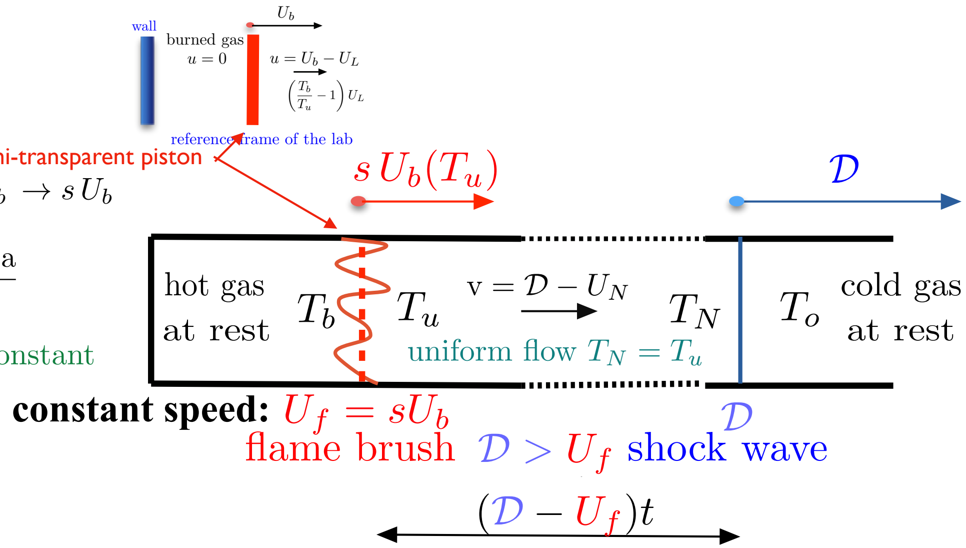
piston effect of the planar flame

piston effect of the wrinkled flame $U_b \to s\,U_b$

folding factor: $s = \dfrac{\text{flame surface area}}{\text{cross section}}$

quasi-steady solution: $U_b = \text{constant}, \quad \mathcal{D} = \text{constant}$

uniform 1-D flow: $s\,(U_b - U_L) = \mathcal{D} - U_N$

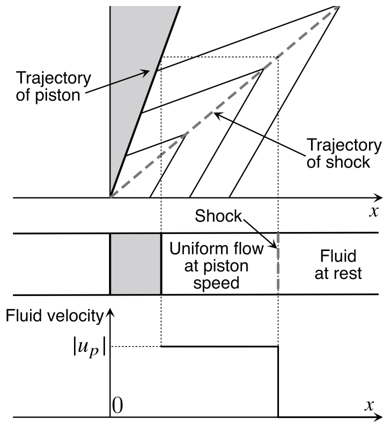


Deshaies and Joulin (1989) *Combust. Flame*, **77**, 201-212

$M_{\text{lead shock}} - 1 \ll 1$

# 1-D self-similar solution of a shock wave generated by a flame brush propagating at a constant velocity $U_{turb}$ from the closed end of a tube

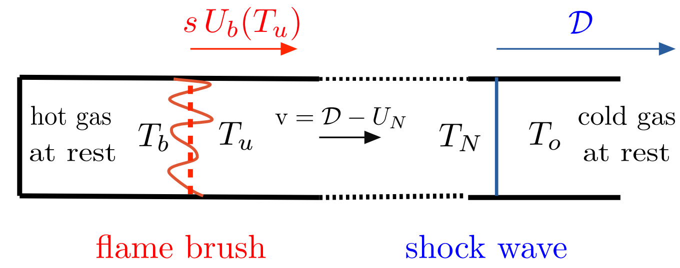
$$U_{turb} = sU_b \qquad s \text{ degree of folding}$$




## Flame $\equiv$ semi-permeable piston

$$\mathrm{v} = s(U_b - U_L) = \left(1 - \frac{\rho_b}{\rho_u}\right) sU_b$$

mass conservation

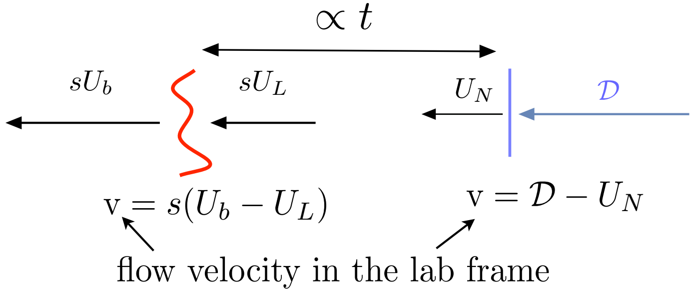



## Lead shock: RH conditions

$$\mathrm{v} = \mathcal{D} - U_N \approx \frac{a_o}{\gamma - 1}\left(\frac{T_N}{T_{uo}} - 1\right)$$

weak shock

$$\approx \frac{a_o}{\gamma - 1}\frac{T_{bo}}{T_{uo}}\left(\frac{T_b}{T_{bo}} - 1\right)$$



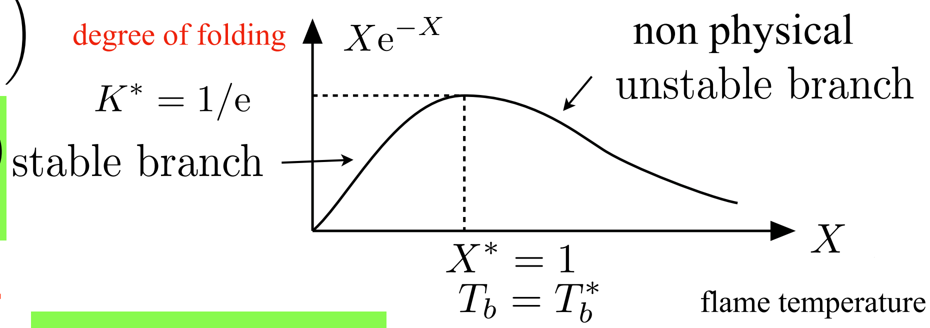

$$T_N = T_u \qquad T_b - T_{bo} = T_N - T_{uo}$$

$$\left(1 - \frac{\rho_b}{\rho_u}\right) sU_b \approx \frac{a_o}{\gamma - 1}\frac{T_{bo}}{T_{uo}}\left(\frac{T_b}{T_{bo}} - 1\right)$$

ZFK flame velocity $\quad U_b/U_{bo} \approx \mathrm{e}^{\frac{E}{2k_B T_{bo}}\left(\frac{T_b - T_{bo}}{T_{bo}}\right)}$

$$\left(1 - \frac{\rho_b}{\rho_u}\right) sU_b(t) = \left(1 - \frac{\rho_b}{\rho_u}\right) sU_b(0)\mathrm{e}^{\frac{E}{2k_B T_b(0)}}\left(\frac{T_b(t) - T_b(0)}{T_b(0)}\right)$$

$$\frac{a_0}{\gamma - 1}\frac{T_{bo}}{T_{uo}}\left(\frac{T_b}{T_{bo}} - 1\right) = \left(1 - \frac{\rho_b}{\rho_u}\right) sU_b(0)\mathrm{e}^{\frac{E}{2k_B T_b(0)}\left(\frac{T_b(t) - T_b(0)}{T_b(0)}\right)}$$




## Nonlinear solution $T_b(s)$ with a turning point

*flame temperature versus degree of folding*

$$X\mathrm{e}^{-X} = K$$

$$X \equiv \frac{E}{2k_B T_{bo}}\left(\frac{T_b}{T_{bo}} - 1\right)$$

flame temperature

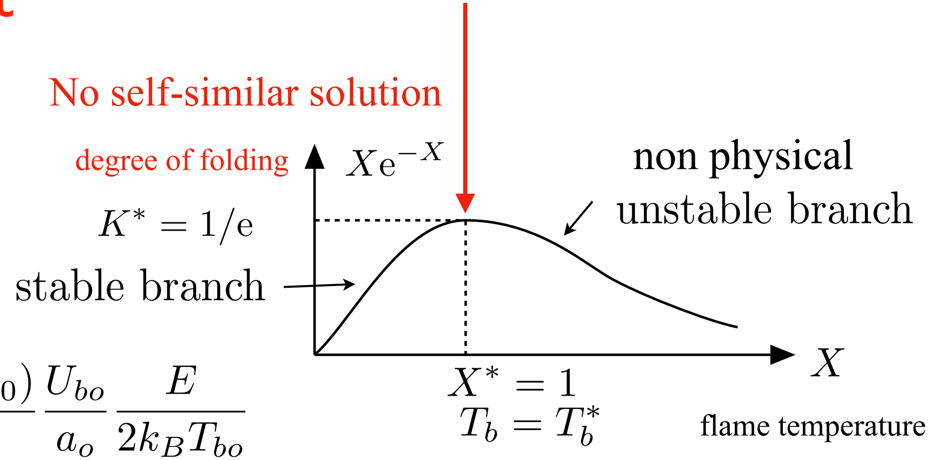
$$K = \kappa s, \qquad \kappa \equiv (\gamma - 1)\frac{(T_{bo} - T_{u0})}{T_{bo}}\frac{U_{bo}}{a_o}\frac{E}{2k_B T_{bo}}$$

degree of folding

# Nonlinear solution $T_b(s)$ with a turning point

*flame temperature versus degree of folding*

$$\boxed{X\mathrm{e}^{-X} = K}$$




$$X \equiv \frac{E}{2k_B T_{bo}}\left(\frac{T_b}{T_{bo}} - 1\right)$$
flame temperature

$$K = \kappa s, \quad \kappa \equiv (\gamma - 1)\frac{(T_{bo} - T_{u0})}{T_{bo}}\frac{U_{bo}}{a_o}\frac{E}{2k_B T_{bo}}$$
degree of folding

Critical folding factor $\quad s^* \approx 10 - 15$

No solution for $K > 1/\mathrm{e}$ i.e when the folding is too large $s > s^*$

If $s \nearrow s^*$ : No more self-similar solution

DDT ?

Numerical solutions of planar flames with a reaction rate multiplied by $s^2 > s^{*2} \approx 10^2 - 10^3$ show a runaway corresponding to DDT (Sivashinsky et al. 2017-2021) (non-physical model !)

Weaknesses of the Deshaies Joulin analysis $\nearrow\!\!\!\!\searrow$ experiments

Weak shock wave

Unsteadiness of the compression wave induced by the self-acceleration of the flame neglected

# DDT in closed tubes

*Recent experiments*

# WORCESTER POLYTECHNIC INSTITUTE

*Massachusetts*


Nolan Dexter-Brown, Jagan Jayachandran



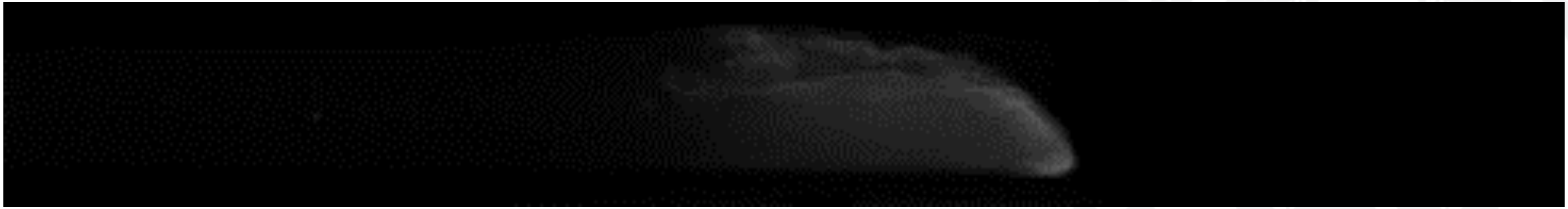
*Combust. Flame 2024*


$$\Delta t = 5 \times 10^{-6} s$$

$$\text{exposure time} = 3 \times 10^{-6} s$$

$$\text{smooth tube:} \quad \phi = 3.8\,\text{cm}, \quad L = 2.4\,\text{m}$$

ethane/oxygen $C_2H_6/O_2$ mixtures



# -2) **Évolution de la vitesse** du front de flamme s'effectue **en 3 étapes avant** l'explosion du point chaud:



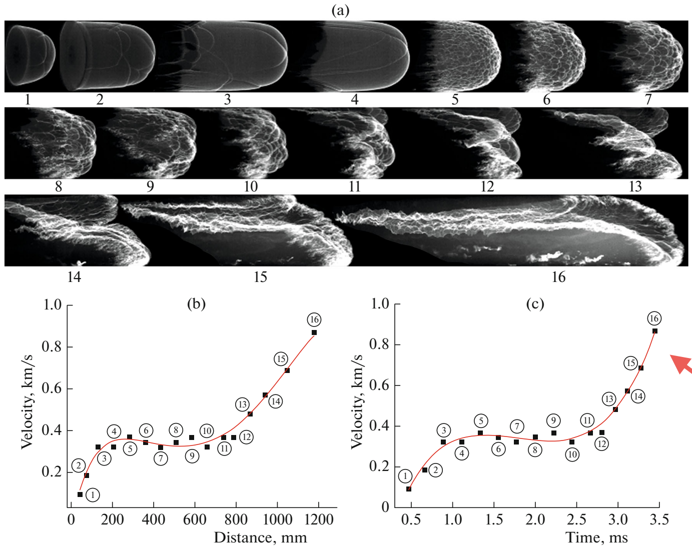



**Fig. 4.** The same as in Fig. 1 at an initial pressure of 16 kPa.

cantly: from two or more diameters to half the tube diameter. The surface of the flame front begins to fragment and a cellular or honeycomb structure is formed on it. Before the flame front, there is a flow of unburned gas in the direction of the front movement, caused by the buoyant action of expanding deflagration products and disturbances generated by the flame that propagate downstream at the speed of sound. Behind the front, the flow caused by rarefaction waves in the deflagration products is directed in the opposite direction. The front, as a discontinuity surface, is subject to gas-dynamic (Darrieus–Landau) and thermal diffusion (Rayleigh–Taylor) instabilities, which also significantly affect the change in the shape of its surface and leads to the appearance of cellular structures. The dimensions of the emerging structure strongly depend on the initial pressure of the gas mixture and sharply decrease with its increase. The flame front's deceleration phase lasts from 1.5 ms for a pressure of 8 kPa to 0.6–0.7 ms for a pressure of 22 kPa. During this time interval, the flame front travels distances of 0.4 and 0.2 m along the tube axis, respectively.

The deceleration phase is followed by a period of time when the flame front propagates at a nearly constant velocity. As can be seen from the images (frames *7–14*, Fig. 1; frames *7–12*, Figs. 2–4; frames *7–9*, Fig. 5), the duration of this stage is from 0.7 ms at an initial pressure of 20.8 kPa to 3 ms at a pressure of 8 kPa. The structure and shape of the flame front during this period barely change. The shape of the deflagration front is close to hemispherical and extended forward by approximately the channel diameter along the tube axis. The surface of the flame front has a pronounced cellular structure, and as the flame moves, the size of the cells begins to grow larger, and their number decreases. They also begin to stretch in the longitudinal direction along the tube axis. This effect is most pronounced at low initial pressures.

Finally, in the fourth stage, the flame front accelerates again. At the same time, its shape and structure undergo significant changes. One of the sections of the flame front begins to move forward sharply in the direction of motion. The authors of [4] found that the upper section of the flame moves forward and explained this as follows: the flame front is the interface between a heavy fresh mixture with a higher density and lighter deflagration products. Under the

P.N. Krivosheyev, A.O. Novitski, O.G. Penyazkov
*Russ.J.Phys.Chem B* (2021) **16** (4) 661-669

$\phi \approx 5$ cm

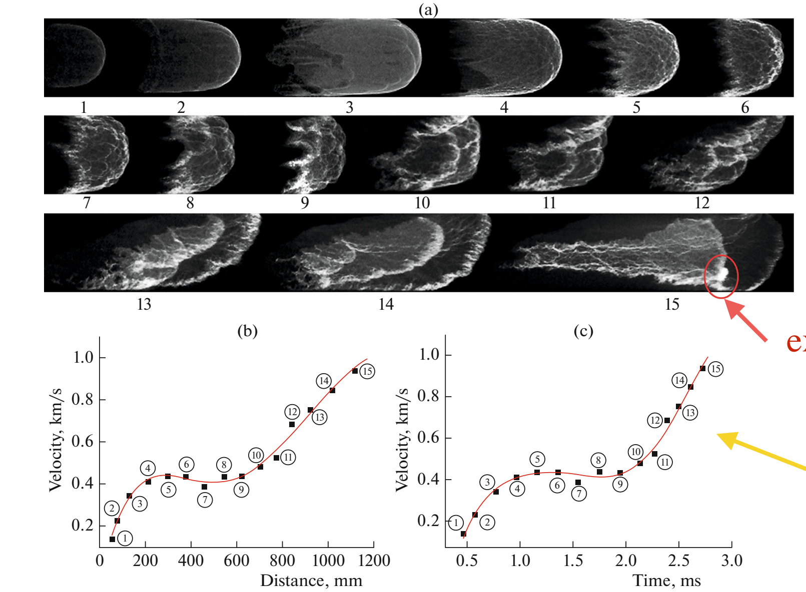
**3 étapes** dans la **vitesse de propagation du front** après allumage ponctuel sur le côté fermé du tube

grande vitesse **juste avant explosion**

$U_f \approx 10^3 m/s$

# -3) **Forte accélération** du front de flamme juste avant l'explosion du point chaud:






explosion d'un point chaud

3ème étape juste avant la transition $(< 10^{-6}s.)$
très forte accélération !
$\Delta V \approx \ 500m/s$ en quelques $10^{-4}s.$

**Fig. 5.** The same as in Fig. 1 at an initial pressure of 20.8 kPa. In frame *15*, the red circle shows the onset of local explosion kernel in the region of the boundary layer.

transition: **brusque changement d'échelle de temps**,

*dernière acceleration*
$10^{-3}s$

explosion
$< 10^{-6}s$

transition à la détonation plane:

action of gravity, the heavy fresh mixture spreads along the bottom of the tube, and the lighter products of deflagration tend to settle in the upper part, pushing the flame front forward. Our observations show that the orientation of the head tongue of the flame front relatively to the tube perimeter is probably random (Figs. 1–5). As the flame front stretches along the tube axis, a significant increase in the deflagration area occurs and its velocity begins to increase drastically. The flame takes on a conical shape strongly elongated along the tube axis, which is described in detail in [82]. In frame *15*, Fig. 5, at an initial mixture pressure of 20.8 kPa, the circle shows the onset of local explosion kernel in the region of the boundary layer in one of the folds of the flame edge encircling the channel walls. The development of this focus leads to the formation of an overcompressed detonation wave, which propagates up and downstream. In experiments at lower initial pressures (8–16 kPa, Figs. 1–4), the process of detonation initiation was no longer within the field of view of the observation system.

CONCLUSIONS

The evolution of the structure and shape of the flame front during the DDT of an acetylene-oxygen mixture in a cylindrical tube 60 mm in diameter was studied by high-speed visualization. The high-speed images show and describe the four characteristic stages of the deflagration process of a gas mixture: at the first stage, the flame accelerates; this is followed by the deceleration stage; then there is the stage of propagation at an almost constant velocity; and, finally, repeated acceleration, during which detonation is formed. Images of the flame shape typical for each stage are given and the dependences of the flame front velocity along the tube axis are determined. The most interesting and insufficiently studied, in our opinion, is the stage of repeated acceleration of the flame, when its speed rapidly increases from 300–400 to 1000–1200 m/s and a detonation wave is formed in front of the flame front (or at its edge). At the same time, during the development of this stage, the shape of the flame front undergoes drastic changes: a conical structure strongly elongated along the tube axis is formed, which is described in detail in [82]. The new



*P.N. Krivosheyev, A.O. Novitski, O.G. Penyazkov (2022)*

-4) Écoulement **laminaire** des gaz frais

-5) Train d'**ondes de choc**

-6) **Nb de Mach** des chocs $M_{shock} \approx 3$ $T < 800K$ **autoallumage impossible**

$T < T_c \approx 1200\,K$



M.A Liberman et al.
*Acta Astronautica* (2010) **67** 688-701

M. Kuznetsov et al.
*Combust. Sci and Tech.* (2010) **182** 1628-1644

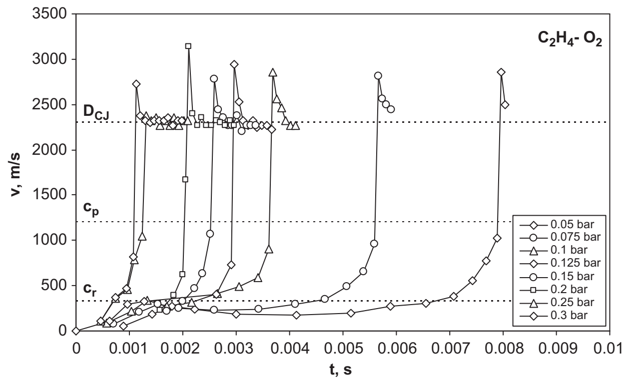
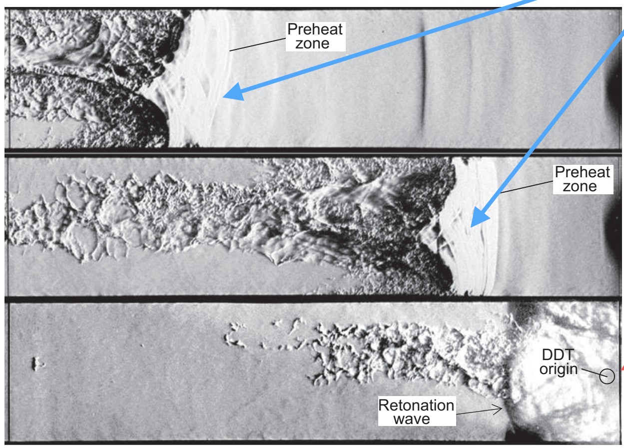



**Fig. 6.** Velocity–time dependence during detonation initiation in ethylene–oxygen mixture.

train d'ondes de compression

Schlieren method :

schlieren




juste avant la transition

juste après la transition

**écoulement laminaire !**

position du point chaud
au moment de l'explosion
**loin de la paroi** (hors couche limite)

**Fig. 7.** Sequence of shadow photographs during the second stage and the transition to detonation in ethylene–oxygen mixture, $P_0$=0.12 bar. Time for the frames from the top is $t$=3.75, 3.85, and 3.95 ms.

$U_f$=589 m/s and $M_{sh}$=2.13. One can see that in contrast to the first stage, the flame generates the compression waves that steepen to shocks almost at the flame front. In the second image, at 3.85 ms, the shocks are coalesced creating a pocket of compressed unreacted gas that appears close ahead to the flame. The width of the preheat zone is 16 mm ($X_f$=1177 mm, $X_{sh}$=1193 mm), the Mach number of the shock is $M_{sh}$=3.03. In the last frame, taken at 3.95 ms, transition to detonation has already occurred and only retonation wave is seen. We want to notice that similar scenario of the preheat zone formation just before DDT can be seen in the Schlieren photographs of earlier experiments [4–6]. This important feature of the flow ahead of the accelerating flame has been overlooked in the previous studies. For the first time the authors of [19,20,22] have pointed that the formation of a preheat zone is the most important feature of the flame dynamics for the mechanism of DDT.

**3. Numerical simulation of DDT in hydrogen–oxygen: formulation of the problem**

The simulations modeled a flame ignited at the closed end and then propagating to the open end of a two-dimensional channel. The computations solved the multi-dimensional, time-dependent, reactive Navier–Stokes equations including the effects of compressible fluid

$T < T_c$

température des gaz frais
trop faible: **autoallumage impossible**

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

**Paul Clavin**

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ECM & CNRS

## Lecture XI
## Initiation of detonations

Part 3 continued : **theoretical analysis of DDT**

# Lectures 11: Initiation of detonation

## Lecture 11-a: Direct initiation

*Flow of burnt gas in spherical CJ detonations*
*Point blast explosions*
*Critical energy*
*Critical dynamics*

## Lecture 11-b: Spontaneous initiation and quenching

*Initiation at high temperature*
*Spontaneous quenching*

## Lecture 11-c: Deflagration to Detonation Transition

*Basic ingredients*
*Shchelkin scenario*
*Thermal feedback*
*Recent experiments*
*Intrinsic DDT mechanism of laminar flames*
*(Theoretical analysis of the runaway of the flame structure)*

My objective is to decipher a DDT mechanism that is **intrinsic to laminar** flame fronts. In a first step, I'll consider **elongated flame fronts in tubes**, and, in a second step, I'll extend the result to **wrinkled or cellular flames.**

Turbulence can promote an early transition but is not an essential DDT mechanism

# Lecture 11-c: Deflagration to Detonation Transition

*Intrinsic DDT mechanism of laminar flames*

*(elongated flame)*

## DDT of elongated flames propagating in **closed** channels

(**laminar** regime)

**Experiments in** $50 \times 50 \text{ mm}^2$ **tubes**:

Liberman et al. (2010) *Acta Astronautica* **67**, 688-701

Kuznetsov et al. (2010) *Combust Sci. Tech.,* **182**, 1628-1644

Dexter-Brown, Jayachandran (2024) *Combust. Flame, 265, 113439*

**Theoretical analyses**

P. Clavin, H. Tofaili. (2021) *Combust. Flame,* **232** 111522

P. Clavin. (2022) *Combust. Flame,* **245**, 112347

P. Clavin, *J. Fluid Mech.* (2023) *vol.* **974**, A46

P. Clavin, *C.R. Acad. Sci. Mécanique* (2023) *vol.* **351**, p. 401-427

P. Clavin, *Eur. Phys. J. Plus* (2025) *vol.* **140**, 258

(**no** assumption of small heat release !)

There is also DDT in **long open ended** channels (not treated in these lectures)

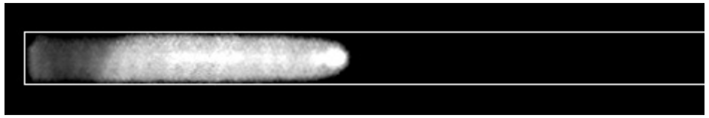
*Experiment:* V. Bykov, et al. (2022) *Combust. Flame,* **238** 111913

*Theory:* P. Clavin, V. Bykov (2024) in preparation

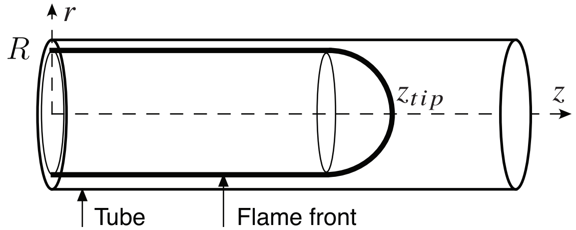
*In addition of the high thermal sensitivity there are two key mechanisms for the DDT of* ***elongated flames*** *in tubes (****laminar*** *regime)*

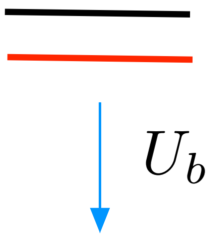
# Self-accelerating elongated flame **(laminar)** of a flame ignited at the **centre of the closed end** of a tube

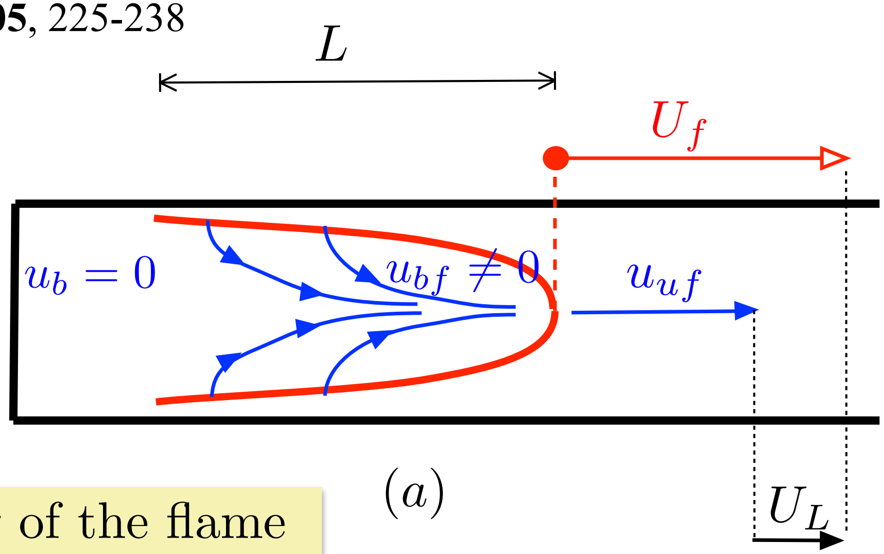
## -1) Back-flow



C Clanet and G. Searby (1996) *Combust. Flame,* **105**, 225-238




Combustion of the lateral wing of the flame
Back-flow of burned gas to the flame tip

-2) Very energetic mixtures: $U_b/U_L = T_b/T_u \approx 10$

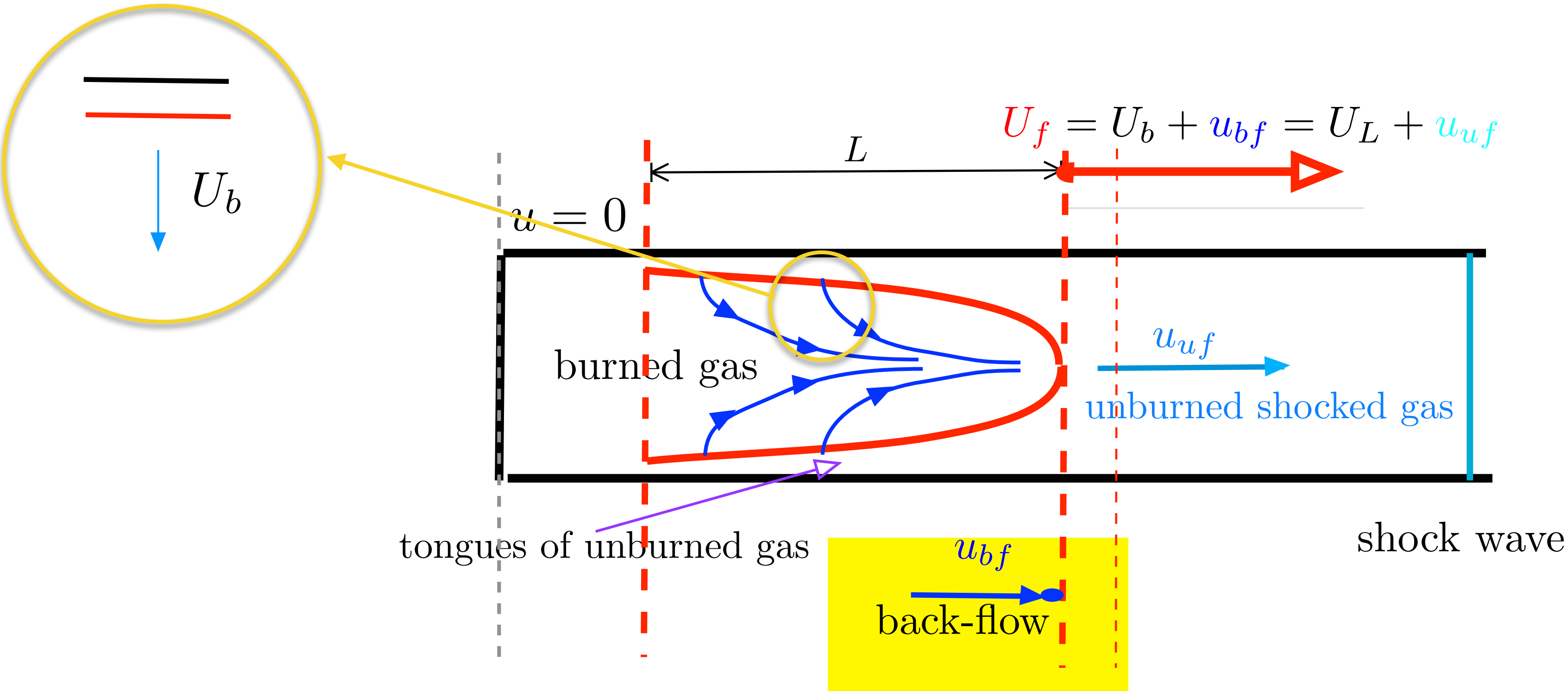
*1st question:*

How to explain a **quasi-sonic** velocity of a flame while its laminar flame velocity is **markedly subsonic ?**

*Réponse for a finger-like flame front:*

the flame on the tip is convected by a **self-induced back-flow**

The curved flow of burned gas is of the type of **a stagnation flow**



# The back flow of burned gas towards the flame tip is well documented by PIV experiments and DNS

Ponizy et al. (2014) *Combust. Flame,* **161** 3051-3062

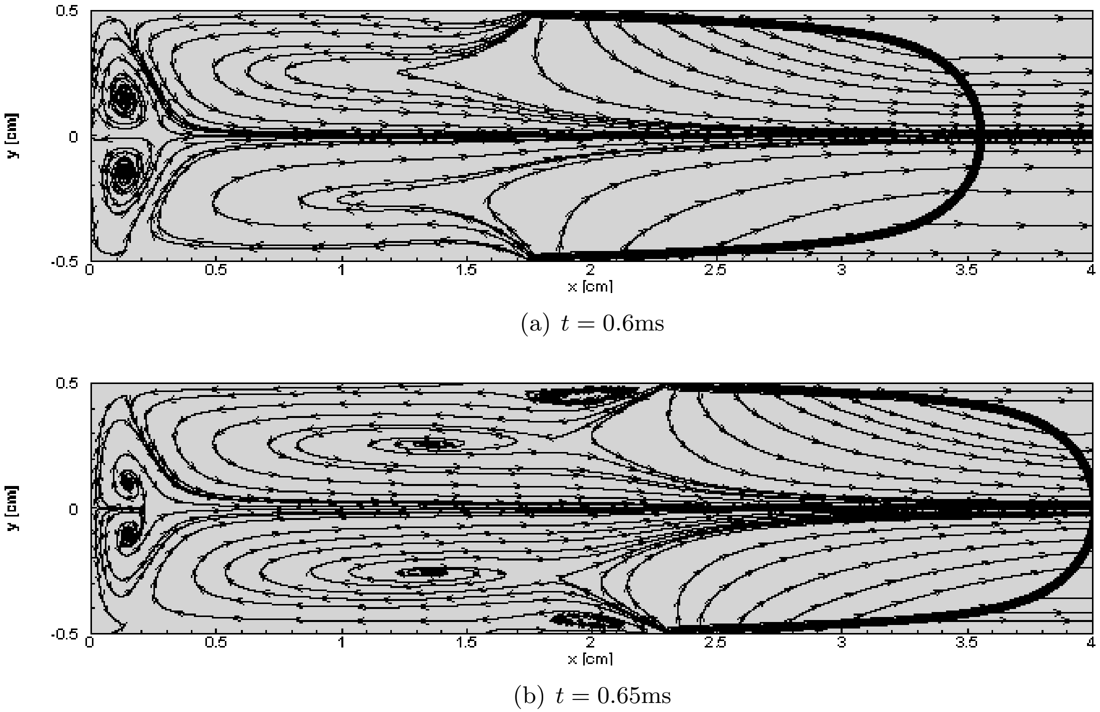
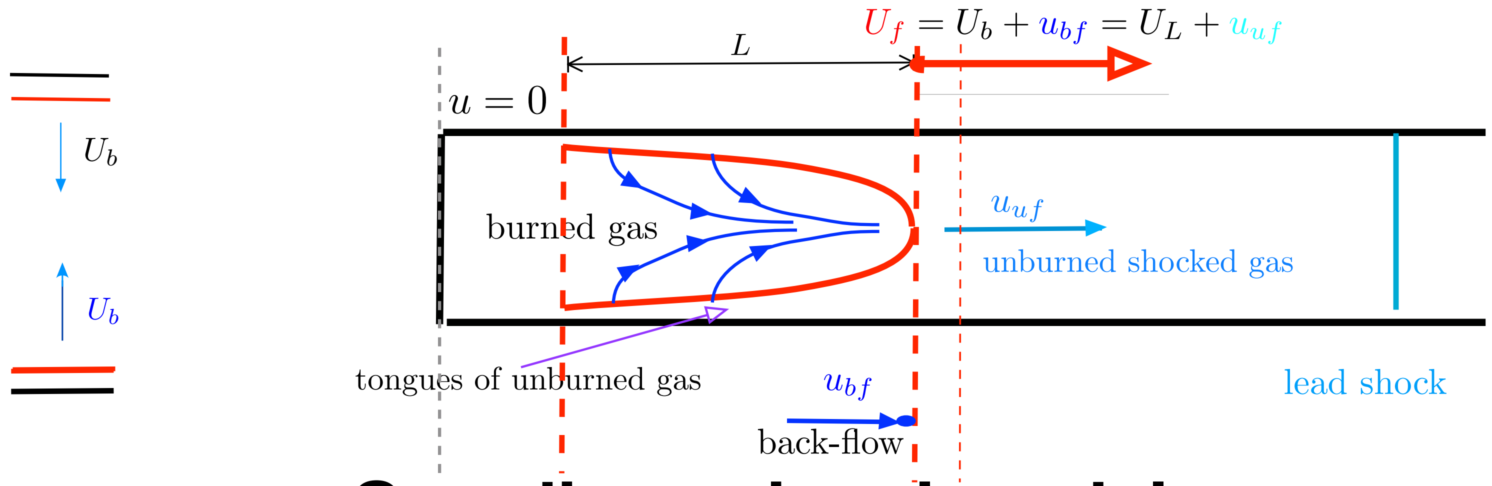



Figure 1: The streamlines and temperature field



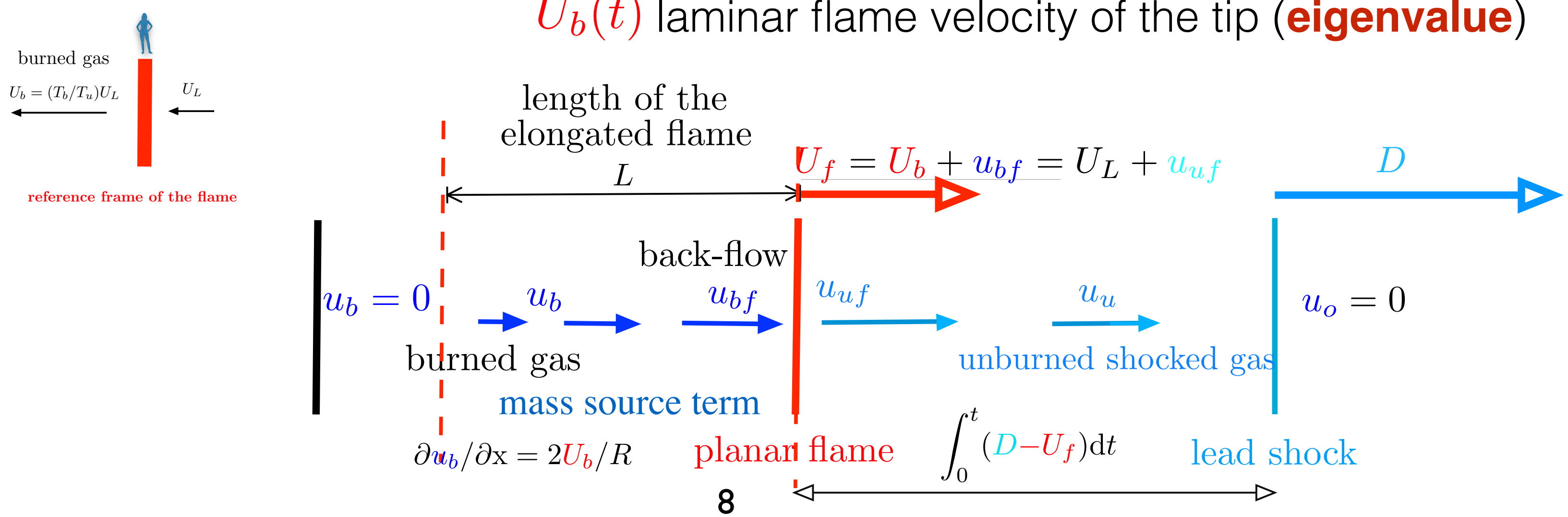

# One-dimensional model

Clavin, Tofaili *Combustion and Flame* **232** (2021)111522

Focus your attention on the **tube axis**, consider the tip as a **planar flame** perpendicular to the axis, and, following C. Clanet and G. Searby (1996) *Combust. Flame,* **105**, 225-238, replace the **curved flow** of burned gas by a **uniform mass source term** $\boxed{2\rho_b U_b(t)/R}$ along the axis of the elongated flame of length $\boxed{L(t)}$

**given function**

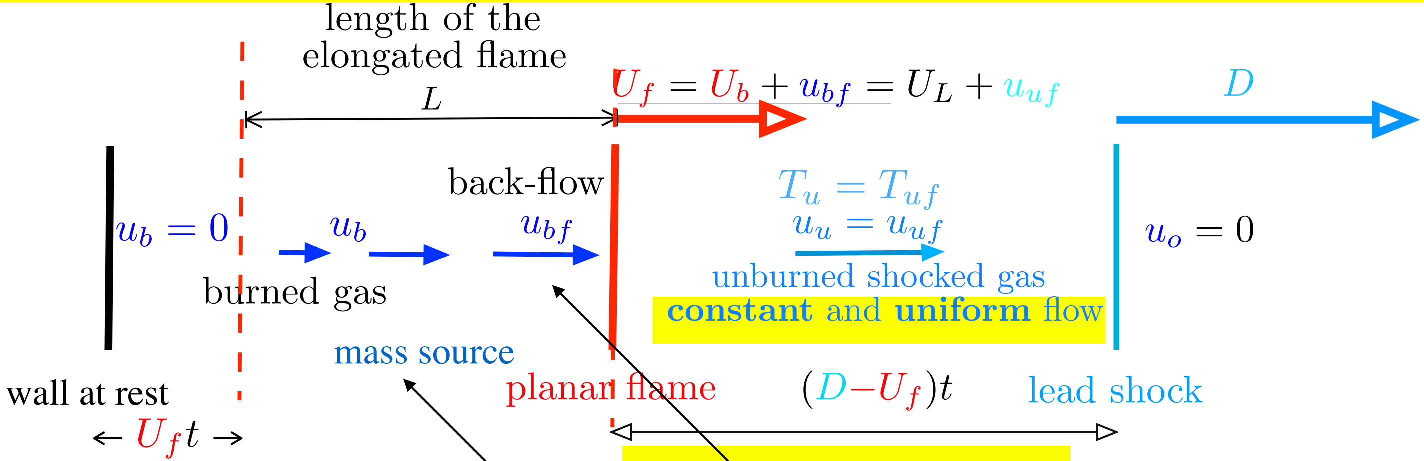
$U_b(t)$ laminar flame velocity of the tip (**eigenvalue**)



# **Self similar** solution for a constant elongation:

$$\mathrm{d}L/\mathrm{d}t = 0 \quad \mathrm{d}U_b/\mathrm{d}t = 0 \quad \mathrm{d}U_f/\mathrm{d}t = 0$$

## **Constant flows and inner structure of the flame in steady state**




elongation parameter $\sigma \approx 10$

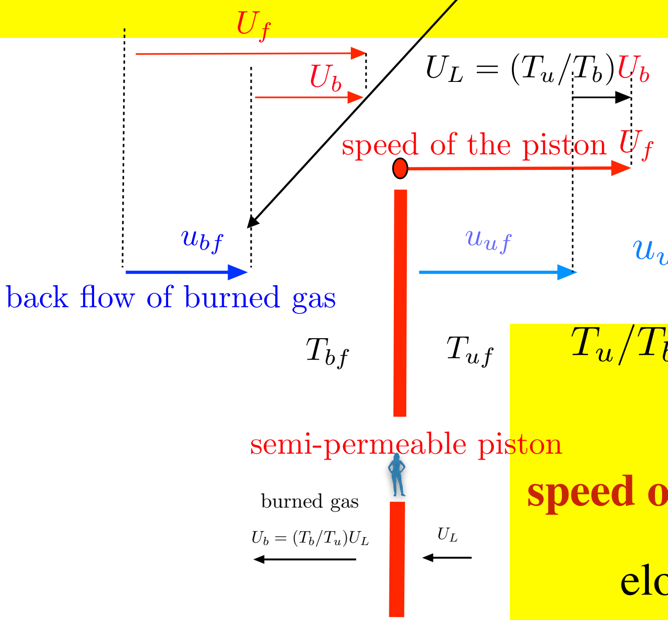
conservation of mass $\quad \dfrac{\partial u_b}{\partial x} = 2\dfrac{U_b}{R}$

**back flow on the flame**

$$u_{bf} \approx U_b \frac{2L}{R}$$

$$U_f = U_b + u_{bf} = \sigma U_b, \qquad \sigma = 1 + 2\frac{L}{R}$$

*Flame acts like a **piston** whose **velocity** increases with the **gas temperature*** $\quad T_u = T_{uf}$




gas expansion across the flame

$$U_L = \frac{T_u}{T_b} U_b$$

$$u_{uf} = U_f - U_L = \left(\sigma - \frac{T_u}{T_b}\right) U_b$$

$T_u/T_b \ll \sigma$ : ***Impermeable piston***

$$u_{uf} \approx U_f = \sigma\, U_b$$

**speed of the piston** $\quad U_f = \sigma\, U_b(T_{bf})$ $\quad$ **Laminar** flame velocity

gas temperature on the piston

elongation parameter $\quad T_{bf} = T_{uf} + q_m/c_p$

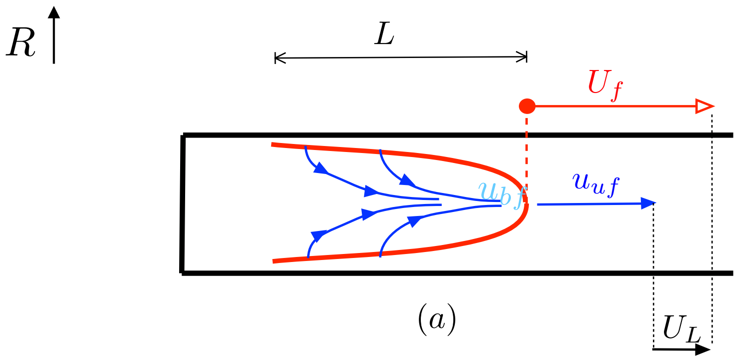
The back-flow of burned gas plays an important role for DDT at the tip of a laminar finger-flame

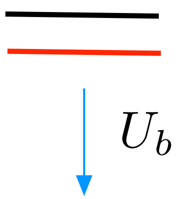
# ONE-DIMENSIONAL PISTON MODEL ON THE TIP

## considering the local solution on the tip as quasi-planar

Clavin, Tofaili *Combustion and Flame* **232** (2021)111522

back-flow of burned gas at the tip $u_{bf}$



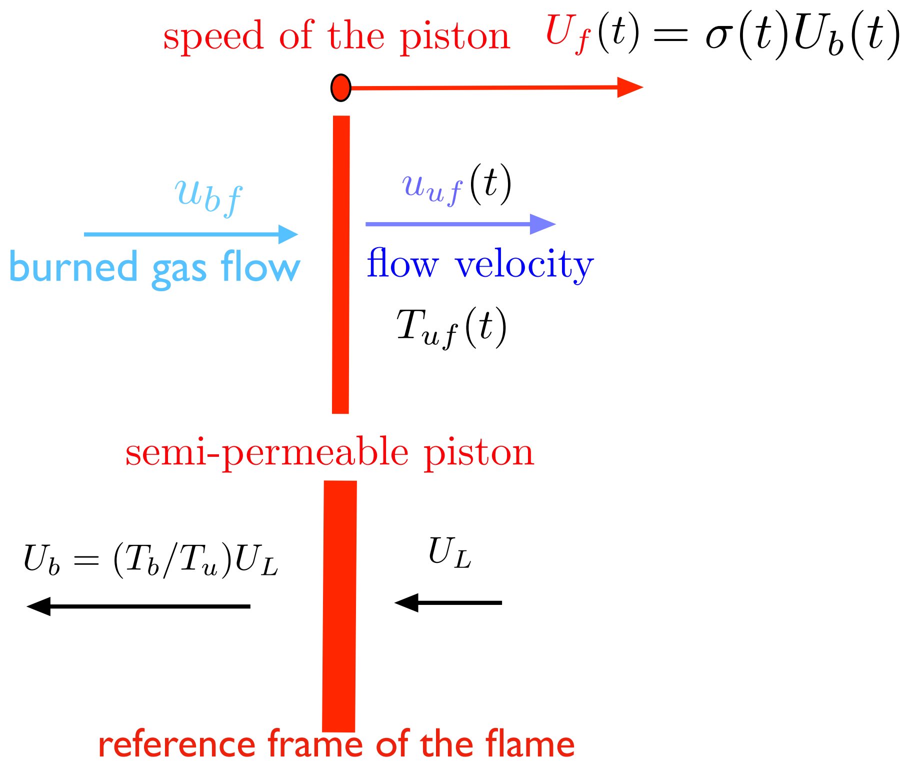

$$\boxed{u_{bf} = 2(L/R)\,U_b}$$

$$U_f = u_{bf} + U_b = \sigma U_b$$

$$\sigma = 2(L/R) + 1$$

$$u_{uf} = U_f - U_L$$

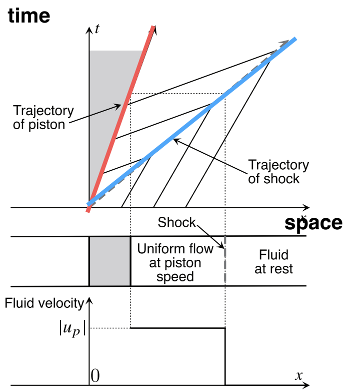
1-D piston model at the tip



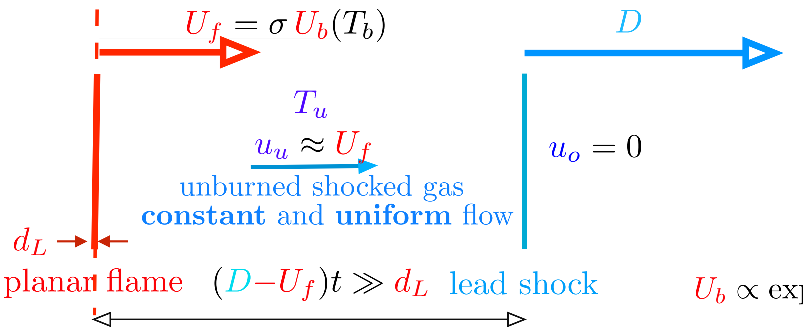

the flows $u_{bf}$ and $u_{uf}$ are functionals of the laminar flame velocity $U_L(T_{uf})$

# Self similar solution for a constant elongation. Piston problem

**Constant velocities**




Attention can be focused only **on the unburned gas flow**




$U_b \propto \exp\left[-\frac{E}{2k_B T_b}\right]$ $\quad \frac{E}{2k_B T_b} \gg 1$

parameter of thermal sensitivity $\quad T_b = T_u + q_m/c_p$

***High thermal sensitivity** of the laminar flame velocity* $\quad \beta \equiv \frac{T_b}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}T_b} \gg 1$ $\quad \frac{\mathrm{d}}{\mathrm{d}T_b} = \frac{\mathrm{d}}{\mathrm{d}T_u}$

***Nonlinear thermal loop** between the **flame** and the **shock:*** ***Two expressions for** $u_u$ **versus** $T_u$* :

-1) **ZFK** on **the flame** : $\quad u_u \approx \sigma U_b(T_b)$ increases **strongly** with $T_u$ through $\beta \gg 1$ (nonlinear relation)

-2) **Rankine-Hugoniot on the shock**: $u_u$ is a quasi-linear function of $T_u$ through the flame Mach number $\varepsilon \equiv U_b/a \ll 1$

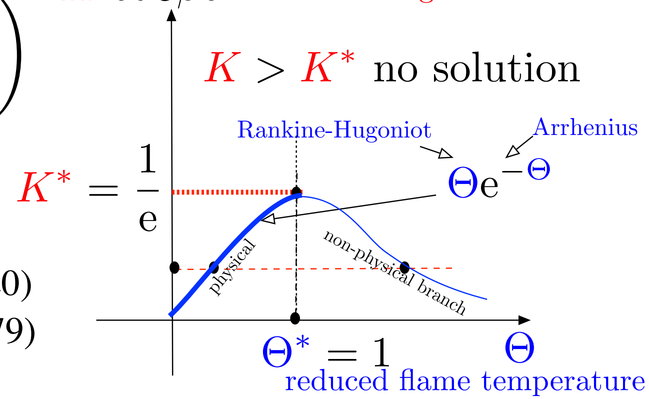
Asymptotic analysis $\quad \beta\varepsilon = O(1)$

Similar to B. Deshaies. & G. Joulin (1989)

***Nonlinear equation for the non-dimensional temperature*** $\quad \Theta \propto \beta\left(\frac{\Delta T_{u,b}}{T_{u,b}}\right)$

***versus the non-dimensional elongation*** $\quad K = \kappa\sigma, \quad \kappa \propto \beta\epsilon$

$K \propto \varepsilon\beta\sigma$ reduced elongation




$RH \qquad$ Arrhenius of $ZFK$

$$\Theta = K \exp\Theta$$

elongation

$$\Theta\exp(-\Theta) = K$$

Semenov eq. for explosion in a vessel (1920)

Joulin-Clavin flame quenching (1976)-(1979)

***Turning point :***

**There is no self-similar solution above a critical elongation** $\quad K > K^* \equiv 1/e$

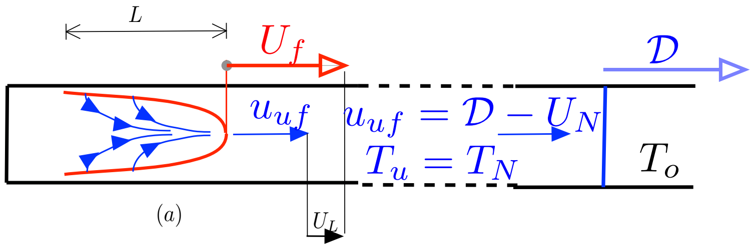
$\Theta = (\beta/2)(T_b/T_{bi} - 1)$

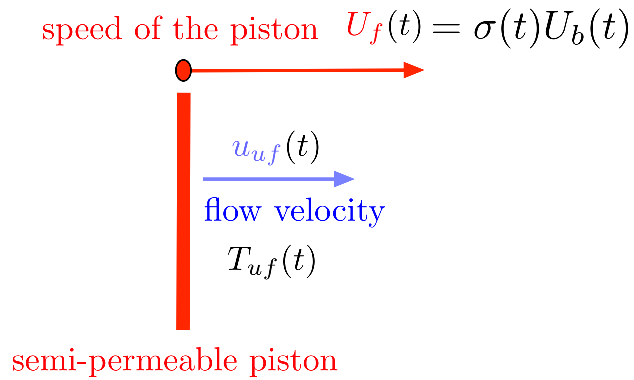
|| ***Transition to detonation ?***

# constant $L$

# Self-similar solution of the one-dimensional piston model

Clavin, Tofaili *Combustion and Flame* **232** (2021)111522




the solution is similar to Deshaies Joulin

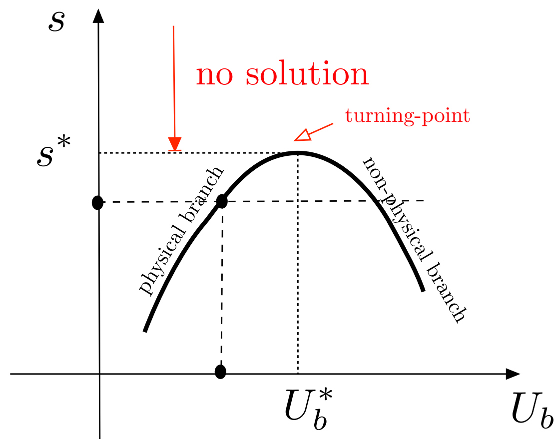
but for a very energetic flame and a back flow

folding factor $\longrightarrow$ elongation $\sigma = 2L/R + 1 \quad U_f = \sigma U_b$

For very energetic mixtures the turning point corresponds to a critical elongation easily accessible by the elongation front usually observed in tubes

$$\frac{U_L(T_u)}{U_L(T_o)} = \left(\frac{T_b}{T_{bo}}\right)^2 \left(\frac{T_u}{T_o}\right)^{3/2} \exp\left[-\frac{E}{2k_B}\left(\frac{1}{T_b} - \frac{1}{T_{bo}}\right)\right], \qquad \text{large } T_b/T_u > 10 \qquad \frac{E}{2k_B T_{bo}} \approx 1$$

$$\frac{U_f(T_u)}{U_f(T_{uo})} = \frac{T_b/T_u}{T_{bo}/T_{uo}} \frac{U_L(T_u)}{U_L(T_{uo})}$$




The critical conditions are in good agreement with DDT of experiments

$$T_u^* \approx 650 \text{ K}, \quad U_L^* \equiv U_L(T_u^*) \approx 40 \text{ m/s}, \quad U_f^* \approx 890 \text{ m/s},$$

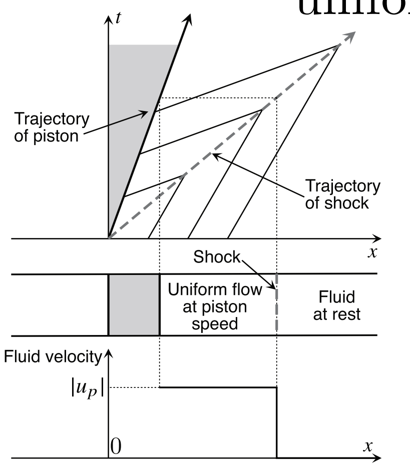
$$M^* \equiv \mathcal{D}^*/a_o \approx 2.5 \qquad L^*/R \approx 1.8$$

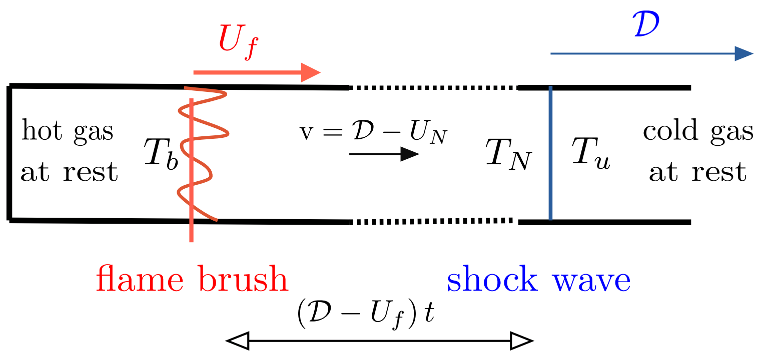
# Limit (weakness) of self-similar solutions of the double-discontinuity model

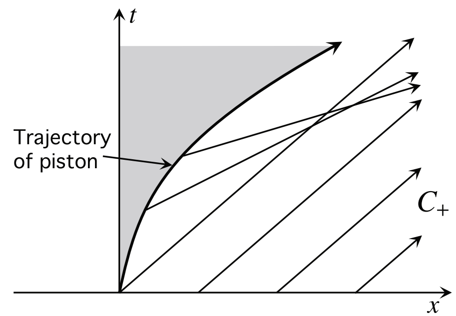
Constant flame velocity
self-similar solution
uniform and constant flow

$\neq$ accelerating flame

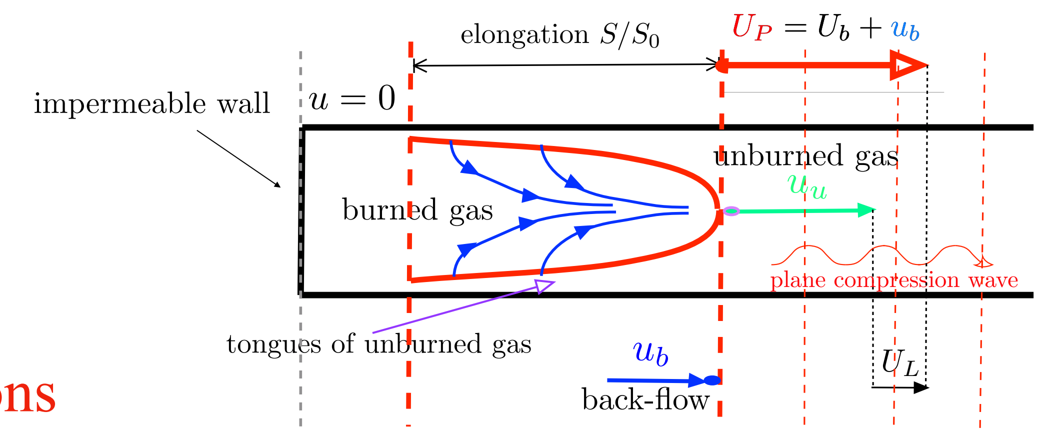
no self similar solution
multiple chocs formation




unsteady flow

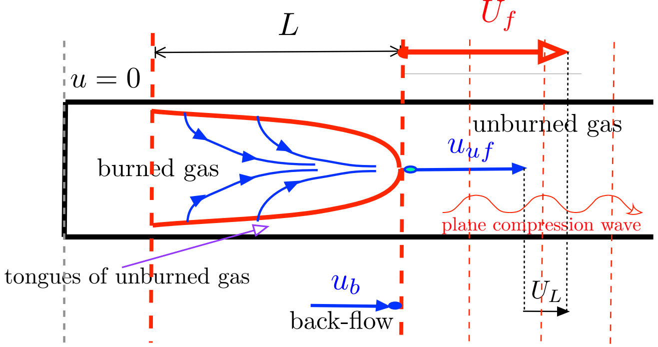



**Unsteady compression waves**
launched ahead of the flame front
by the **acceleration** of the tip
when the elongation increases
is an essential mechanism of DDT
that is **ignored** by the self-similar solutions

flame elongation

$\mathrm{d}L(t)/\mathrm{d}t > 0 \quad L(t) \nearrow \quad U_f(t) \nearrow$

$\sigma = 2(L/R) + 1 \qquad \sigma(t) \nearrow$




unsteady compression waves ahead of the flame producing the thermal feedback

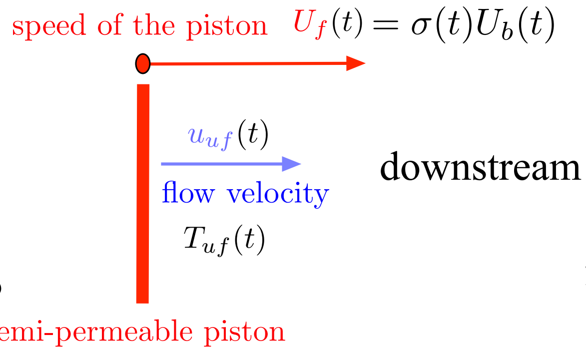
# One-dimensional piston model beyond self similarity

Clavin, P. (2022) *Combust. Flame*, **245**, 112347

$$U_f = [2(L/R) + 1]U_b \qquad U_f(t) = \sigma(t)U_b(t)$$

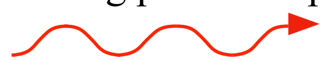
$$U_L = (T_u/T_b)U_b \qquad \sigma(t) \equiv 2\frac{L(t)}{R} + 1$$

$$u_{uf} = U_f - U_L$$




$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} > 0$$

downstream running plane compression wave



flow of unburned gas

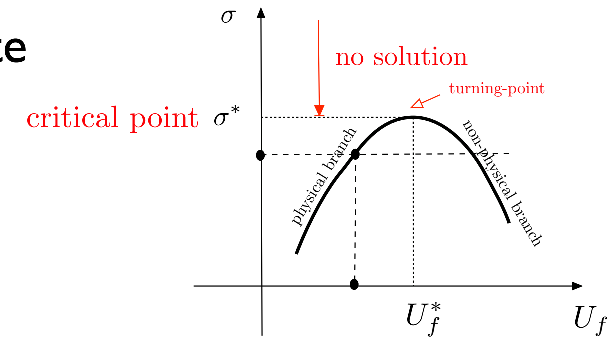
energetic mixture $\quad T_u/T_b < 1/10, \quad 2(L^*/R) + 1 \approx 5$

$$u_{uf} \approx U_f$$

the piston is quasi-impermeable near the turning point

Start the increase in elongation from an initial state in steady state constituted by a self-similar solution close to the turning point

$$\sigma(t) = \sigma(0)[1 + t/t_e]$$



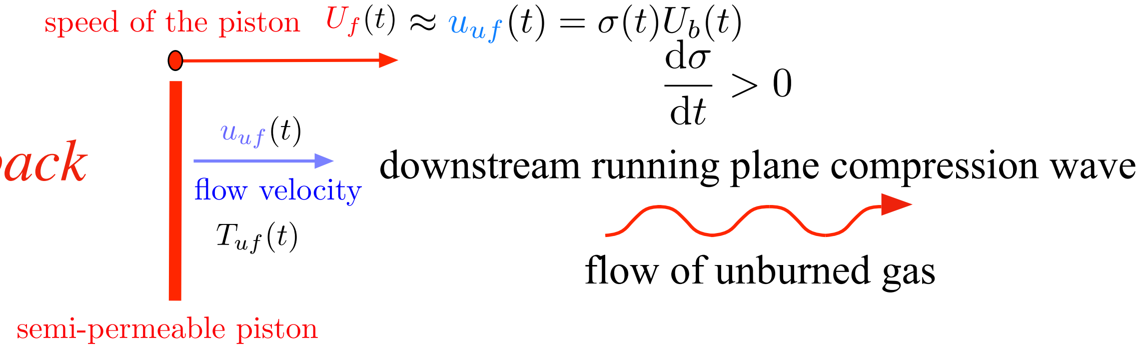

The initial lead shock is assumed far ahead the flame

$$\frac{u_{uf}^* - u_{uf}(0)}{a_{uf}(0)} \ll 1 \quad \Rightarrow \quad \text{compression waves} \approx \text{downstream running acoustic waves}$$

# Beyond self similarity

## *Acceleration induced thermal feedback*

Clavin, P. (2022) *Combust. Flame,* **245**, 112347




$\dfrac{u_{uf}(t) - u_{uf}(0)}{a_{uf}(0)} \ll 1$ compression waves $\approx$ downstream running acoustic waves

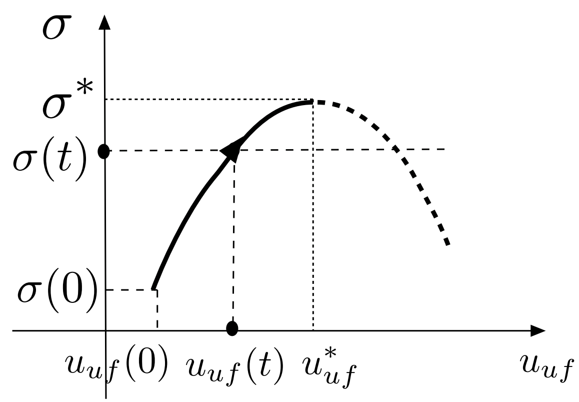
Riemann: $$\boxed{\frac{T_{uf}(t)}{T_{uf}(0)} - 1 \approx (\gamma - 1)\left[\frac{u_{uf}(t) - u_{uf}(0)}{a_{uf}(0)}\right]}$$

$\sigma(t) = \sigma(0)[1 + t/t_e]$ $\quad 1/t_e \ll 1/\tau_L$ flame structure in quasi-steady state

Z FK: $$\frac{U_b(t)}{U_b(0)} = \exp\left[\frac{1}{2}\frac{E}{k_B T_b(0)}\left(\frac{T_{uf}(t) - T_{uf}(0)}{T_b(0)}\right)\right]$$

$u_{uf}(t) = \sigma(t) U_b(t) \quad \Rightarrow$ **new** turning point of the function $u_{uf}(\sigma)$
at a critical elongation $\sigma = \sigma^*, \quad u_{uf} = u^*_{uf}$

$\boxed{\sigma(t) \nearrow t}$ $\qquad \sigma(t^*) = \sigma^* \qquad u_{uf}(t^*) = u^*_{uf}$




there is no more **dynamical solution** after a finite time

$$\sigma > \sigma^* \quad \Leftrightarrow \quad t > t^*$$

What happens when approaching $\sigma^*$?

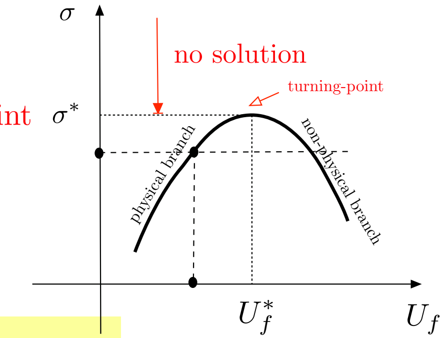
Strong unsteady effects: finite time singularity of the dynamical solution !

(as shown in the rest of the lecture)

## Dynamics close to the critical point when the flame and the burned gas are in steady state, retaining only **unsteadiness in the unburned gas**:

strongly unsteady effects of the compression wave develop when approaching the critical point

generic expression for $U_f(\sigma)$ : $\dfrac{\sigma-\sigma^*}{\sigma^*} \propto -\dfrac{(U_f-U_f^*)^2}{U_f^{*2}}$ $\quad U_f \approx u_{uf}$




elongation growth rate

$$\sigma(t)=\sigma(0)[1+t/t_e]$$

critical velocity

critical time $\quad t-t^* \to 0^-$ : $\dfrac{(U_f^*-U_f)}{U_f^*}=\sqrt{\dfrac{t^*-t}{t_e^*}} \quad \Rightarrow \quad \dfrac{t_e^*}{U_f^*}\dfrac{\mathrm{d}U_f}{\mathrm{d}t}=\sqrt{\dfrac{t_e^*}{t^*-t}},$ $\quad t_e^* \propto t_e$ reference time

finite-time singularity of the flame acceleration $\quad \displaystyle\lim_{(t-t^*)\to 0^-}\frac{\mathrm{d}U_f}{\mathrm{d}t}=\infty \quad \forall\ t_e^*$

## What about the flow field ?

$$\frac{\partial u}{\partial t}+\left(\frac{\gamma+1}{2}u+a(0)\right)\frac{\partial u}{\partial x}=0 \qquad x=X_f(t):\ u=U_f(t)=u_{uf}(t)$$

Riemann: Burgers equation $\qquad \mathrm{d}X_f/\mathrm{d}t=U_f(t) \qquad U_f(t)-U_f(0)\ll a(0)$

$$1-\frac{u(x,t)}{u^*_{uf}} \propto \sqrt{\frac{(t^*-t)}{t_e^*}+\frac{1}{\kappa_P}\frac{(x-X_f(t))}{a(0)t_e^*}} \qquad \kappa_P \equiv 1+U_f^*/a(0)$$

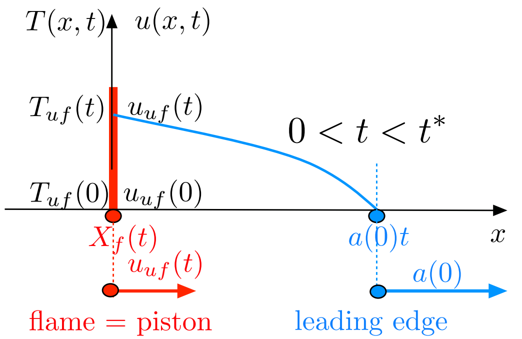
$$\left.\frac{1}{u^*_{uf}}\frac{\partial u}{\partial x}\right|_{x=X_f(t)}=-\frac{1}{2a(0)t_e^*}\frac{1}{\sqrt{(t^*-t)/t_e^*}}$$

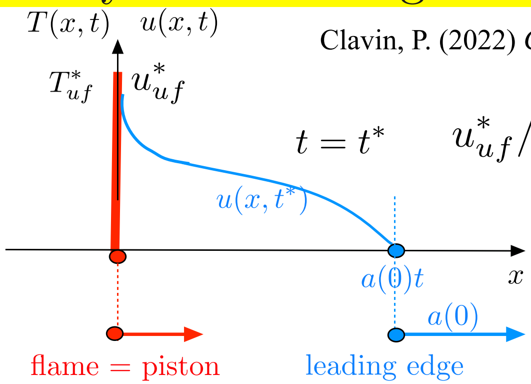
Clavin, P , Tofaili.H. (2022) *Combust. Flame,* **232,** 111522

Clavin, P , Champion,.M. (2023) *Combust. Sci. Technol,* **195,** (15) pp. 3663-3694

singularity of the flow gradient on the piston at $t=t^*$

Clavin, P. (2022) *Combust. Flame,* **245**, 112347




$u^*_{uf}/a(0) \approx [2\beta(\gamma-1)]^{-1}$

**ONSET OF A SHOCK IN THE FLAME STRUCTURE: DDT ?**

## **Details of the calculation**: Piston-driven flow field in a gas initially at uniform velocity (simple wave)

$$\mathrm{d}X_f/\mathrm{d}t = U_f(t) \qquad\qquad t<0, \quad x>0: \quad u(x,t) = U_f(0)$$

*the piston velocity is a given function* *initiai condition*

### General method of solution before the shock formation (isentropic pb)

*bc at* $x \to \infty: \ u = U_f(0),\ a = a(0)$

Riemann invariant. Simple wave

$$J_- \equiv u - \frac{2a}{\gamma-1} = U_f(0) - \frac{2a(0)}{\gamma-1} \qquad \Rightarrow \qquad a = a(0) + \frac{\gamma-1}{2}[u - U_f(0)]$$

$$\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]\left[u + \frac{2a}{\gamma-1}\right] = 0 \qquad J_+ \equiv u + 2\frac{a}{\gamma-1} = 2\left[u + \frac{a(0)}{\gamma-1}\right] - U_f(0)$$

$$u + a = \frac{\gamma+1}{2}u + a(0) - \frac{\gamma-1}{2}U_f(0)$$

$$\frac{\partial u}{\partial t} + \left[\frac{\gamma+1}{2}u + a(0) - \frac{\gamma-1}{2}U_f(0)\right]\frac{\partial u}{\partial x} = 0$$

$$u + a = \frac{\gamma+1}{2}u + b \qquad b \equiv a(0) - \frac{\gamma-1}{2}U_f(0)$$

Burgers equation

$$\frac{\partial u}{\partial t} + \left[\frac{\gamma+1}{2}u + b\right]\frac{\partial u}{\partial x} = 0$$

Formal solution: $f(u) = x - \left[\frac{\gamma+1}{2}u + b\right]t$ where the unknown function $f(u)$ is determined by the boundary conditions

bc on the piston: $f(U_f) = X_f - \left[\frac{\gamma+1}{2}U_f + b\right]t \quad \Leftarrow \quad x = X_f(t): \quad u = U_f(t),$

$X_f(U_f)$ $\qquad t(U_f)$

$$\mathrm{d}X_f/\mathrm{d}t = U_f, \quad X_f(0) = 0: \quad X_f(t) = \int_0^t U_f(t')\mathrm{d}t'$$

$X_f(0) = 0.$ *bc away from the piston weak discontinuity* $X_f(t) = [a(0) + U_f(0)]t: \quad u = U_f(0) \ \Rightarrow \ f(U_f(0)) = 0$

### Near the turning point:

$$t - t^* \to 0^-: \frac{(U_f^* - U_f)}{U_f^*} = \sqrt{\frac{t^*-t}{t_e^*}}, \qquad \frac{t^*-t}{t_e^*} = \left(1 - \frac{U_f}{U_f^*}\right)^2, \quad \frac{X_f(t) - X_f^*}{U_f^*} = -(t^*-t) + \frac{2}{3}\frac{1}{\sqrt{t_e^*}}(t^*-t)^{3/2}$$

$$\frac{t^*}{t_e^*} < 1, \quad \frac{U_f(0)}{U_f^*} = 1 - \sqrt{\frac{t^*}{t_e^*}}, \qquad -\frac{X_f^*}{U_f^* t_e^*} = -\frac{t^*}{t_e^*} + \frac{2}{3}\left(\frac{t^*}{t_e^*}\right)^{3/2}, \quad \frac{X_f - X_f^*}{U_f^* t_e^*} = -\left(1 - \frac{U_f}{U_f^*}\right)^2 + \frac{2}{3}\left(1 - \frac{U_f}{U_f^*}\right)^3,$$

$$f(U_f) = X_f(U_f) - \left[\frac{\gamma+1}{2}U_f + b\right] t(U_f) \qquad U_f \to u$$

$$\frac{f(u)}{U_f^* t_e^*} = -\left(1-\frac{u}{U_f^*}\right)^2 + \frac{X_f^*}{U_f^* t_e^*} - \left[\frac{\gamma+1}{2}\left(\frac{u}{U_f^*}-1\right) + \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\right]\left[\frac{t^*}{t_e^*} - \left(1-\frac{u}{U_f^*}\right)^2\right]$$

$$= \frac{X_f^*}{U_f^* t_e^*} - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\frac{t^*}{t_e^*} + \frac{\gamma+1}{2}\left(1-\frac{u}{U_f^*}\right)\frac{t^*}{t_e^*} + \left(\frac{\gamma-1}{2}+\frac{b}{U_f^*}\right)\left(1-\frac{u}{U_f^*}\right)^2 + O\left(\left[1-\frac{u}{U_f^*}\right]^2\right)$$

$$f(u) = x - \left[\frac{\gamma+1}{2}u + b\right] t$$

$$\frac{f(u)}{U_f^* t_e^*} = \frac{x-X_f^*}{U_f^* t_e^*} + \frac{X_f^*}{U_f^* t_e^*} - \frac{\gamma+1}{2}\left(\frac{u}{U_f^*}-1\right)\left(\frac{t-t^*}{t_e^*}+\frac{t^*}{t_e^*}\right) - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\frac{t}{t_e^*}$$

$$= \frac{x-X_f^*}{U_f^* t_e^*} + \frac{X_f^*}{U_f^* t_e^*} - \frac{\gamma+1}{2}\left(\frac{u}{U_f^*}-1\right)\frac{t^*}{t_e^*} - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\left(\frac{t-t^*}{t_e^*}\right) - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\frac{t^*}{t_e^*} + O\left(\frac{(t^*-t)^{3/2}}{t_e^{*3/2}}\right)$$

$$= \frac{x-X_f^*}{U_f^* t_e^*} - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\left(\frac{t-t^*}{t_e^*}\right) + \frac{X_f^*}{U_f^* t_e^*} - \frac{\gamma+1}{2}\left(\frac{u}{U_f^*}-1\right)\frac{t^*}{t_e^*} - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\frac{t^*}{t_e^*} + O\left(\frac{(t^*-t)^{3/2}}{t_e^{*3/2}}\right)$$

## putting together

$$\left(\frac{\gamma-1}{2}+\frac{b}{U_f^*}\right)\left(1-\frac{u}{U_f^*}\right)^2 \approx \frac{x-X_f^*}{U_f^* t_e^*} - \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\left(\frac{t-t^*}{t_e^*}\right)$$

$$\left(1-\frac{u}{U_f^*}\right) \propto \sqrt{\frac{x-X_f^*}{U_f^* t_e^*} + \left(\frac{\gamma+1}{2}+\frac{b}{U_f^*}\right)\left(\frac{t^*-t}{t_e^*}\right)}$$

Limitation of the analysis near the critical point : strong acceleration

$\Rightarrow$ neither the flame structure nor the burned gas flow are in quasi-steady state

## Fully unsteady analysis of the dynamics when approaching the critical point by a perturbation analysis for a small elongation rate

P. Clavin, *J. Fluid Mech.* (2023) *vol.* **974**, A46

P. Clavin, *C.R. Acad. Sci. Mécanique Paris* (2023) *vol.* **351**, p. 401-427

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Non dimensional equations are obtained with the initial state of the burned gas used as reference

$$\tau \equiv t/t_L(0) \quad \text{where} \quad t_L(0) \equiv d_L(0)/U_b(0), \qquad \xi \equiv (\mathrm{x} - X_f)/d_L(0),$$

$$r \equiv \frac{\rho}{\rho_b(0)}, \quad \mathrm{v} \equiv \frac{u}{U_b(0)}, \quad \pi \equiv \frac{p}{p(0)}, \quad \theta \equiv \frac{T}{T_b(0)}, \quad \boxed{\epsilon \equiv \frac{t_L(0)}{t_e},} \quad \frac{\sigma(\tau)}{\sigma(0)} = 1 + \epsilon\tau, \quad \boxed{\varepsilon \equiv \frac{U_b(0)}{a_u(0)}}$$

$\theta \neq (T - T_u)/(T_b - T_u)$

definition of $\epsilon$ and $\varepsilon$

small elongation rate $\epsilon \ll 1$ elongation small Mach number $\varepsilon \ll 1$

Mass-weighted coordinate: $(\xi, \tau) \to (z, \tau)$, $\quad z \equiv \int_0^{\xi} r(\xi', \tau)\mathrm{d}\xi'$, $\quad \frac{\partial}{\partial \xi} = r\frac{\partial}{\partial z}$

Asymptotic analysis with the ZFK model of flame: $\varepsilon \ll \epsilon \ll 1, \quad (\gamma - 1)\beta\varepsilon\sigma(0) = O(1), \quad \pi = 1 + \varepsilon\pi_1(\varepsilon z, \tau)$

up to order $\varepsilon$ included for retained the pressure effect $\quad \beta \equiv E/k_B T_b(0) \to \infty$ multiple scale problem

small Mach number small elongation rate

$\beta\varepsilon = O(1)$

large activation energy small Mach nb

Distinguished limit

Equations outside the reaction sheet for $\theta, \pi_1, \mathrm{v}, \psi$ $\quad \frac{D(1/\rho)}{Dt} = \frac{\partial u}{\partial z} \quad \& \quad 1/\rho = (c_p - c_v)T/p$

mass+energy+ideal gas $\quad \frac{\partial \mathrm{v}}{\partial z} = [1 - \varepsilon\pi_1]\frac{\partial^2\theta}{\partial z^2} - \varepsilon\frac{1}{\gamma}\theta\frac{\partial \pi_1}{\partial \tau} + O(\varepsilon^2)$

momentum $\quad \left[\frac{\partial \mathrm{v}}{\partial \tau} - m(\tau)\frac{\partial \mathrm{v}}{\partial z} - \frac{\partial^2 \mathrm{v}}{\partial z^2}\right] = -\frac{1}{\gamma}\frac{1}{\varepsilon}\frac{\partial \pi_1}{\partial z}$ $\quad m(\tau) \equiv$ reduced mass flux across the reaction sheet

$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial \mathrm{x}} + \frac{\partial}{\partial \mathrm{x}}\left[\mu\frac{\partial u}{\partial \mathrm{x}}\right]$$

species $\quad \left[\frac{\partial \psi}{\partial \tau} - m(\tau)\frac{\partial \psi}{\partial z} - \frac{\partial^2 \psi}{\partial z^2}\right] = 0, \qquad \psi(z, \tau) \in [0,\ 1]$

energy $\quad \left[\frac{\partial \theta}{\partial \tau} - m(\tau)\frac{\partial \theta}{\partial z} - \frac{\partial^2 \theta}{\partial z^2}\right] = \varepsilon\frac{(\gamma - 1)}{\gamma}\theta\left[\frac{\partial \pi_1}{\partial \tau} - \cancel{m(\tau)\frac{\partial \pi_1}{\partial z}}\right], \qquad \frac{\partial \pi_1}{\partial z} = O(\varepsilon)$

Boundary conditions

*hyperbolic problem in the external zone* $\quad z \to \pm\infty :$ initial conditions $\qquad z = 0 : \ \mathrm{v} = \mathrm{v}_b = \sigma(0)[1 + \epsilon\tau]\, m \ +$ ZFK jump conditions for $\theta, \psi$

# Formulation. Equations outside the reaction zone

*(details of the calculation)*

$$\mathrm{m}(\xi,\tau) = r(\xi,\tau)\big[\mathrm{V}_f(\tau) - \mathrm{v}(\xi,\tau)\big] > 0, \quad \mathrm{V}_f \equiv U_f(\tau)/U_b(0), \quad r \equiv \pi/\theta$$

mass flux across the reaction sheet $\xi = 0:$ $\boxed{m(\tau) \equiv \mathrm{m}(0,\tau) = r(0,\tau)\big[\mathrm{V}_f(\tau) - \mathrm{v}(0,\tau)\big]}$

change of variables:

$(\xi,\tau) \to (z,\tau)$ $\quad z \equiv \int_0^\xi r(\xi',\tau)\mathrm{d}\xi', \quad \frac{\partial}{\partial \xi} = r\frac{\partial}{\partial z} = \frac{\pi}{\theta}\frac{\partial}{\partial z} \qquad \frac{\partial}{\partial \tau}\Big|_\xi = \frac{\partial}{\partial \tau}\Big|_z + \left[\int_0^\xi \frac{\partial r(\xi',\tau)}{\partial \tau}\mathrm{d}\xi'\right]\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau}\Big|_z + [\mathrm{m}(\xi,\tau) - m(\tau)]\frac{\partial}{\partial z}$

Mass weighted coordinate

$$\frac{D}{D\tau} = \frac{\partial}{\partial \tau}\Big|_\xi + (\mathrm{v} - \mathrm{V}_f(\tau))\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \tau}\Big|_z - m(\tau)\frac{\partial}{\partial z}$$

mass: $\quad \frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial \mathrm{x}} \quad \frac{\partial r}{\partial \tau} = \frac{\partial\,\mathrm{m}(\xi,\tau)}{\partial \xi} \quad \frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{\partial u}{\partial \mathrm{x}} \quad \frac{D(1/\rho)}{Dt} = \frac{1}{\rho}\frac{\partial u}{\partial \mathrm{x}}$ $\quad \boxed{\frac{\partial \mathrm{v}}{\partial z} = \left[\frac{\partial}{\partial \tau} - m(\tau)\frac{\partial}{\partial z}\right]\frac{\theta}{\pi},}$

$$\frac{\partial r}{\partial \tau} - \mathrm{V}_f(\tau)\frac{\partial r}{\partial \xi} = -\frac{\partial[r(\xi,\tau)\mathrm{v}(\xi,\tau)]}{\partial \xi}$$

momentum: $\quad \rho\frac{Du}{Dt} = -\frac{\partial p}{\partial \mathrm{x}} + \frac{\partial}{\partial \mathrm{x}}\left[\mu\frac{\partial u}{\partial \mathrm{x}}\right] \qquad \pi \equiv \frac{p}{p_f(0)} = 1 + \varepsilon\pi_1(\varepsilon\xi,\tau) \qquad \mathrm{Pr} = 1$ $\quad \boxed{\left[\frac{\partial \mathrm{v}}{\partial \tau} - m(\tau)\frac{\partial \mathrm{v}}{\partial z} - \frac{\partial^2 \mathrm{v}}{\partial z^2}\right] = -\frac{1}{\gamma}\frac{1}{\varepsilon}\frac{\partial \pi_1}{\partial z}}$

energy: $\quad \rho c_p\frac{DT}{Dt} = \frac{Dp}{Dt} + \frac{\partial}{\partial \mathrm{x}}\left[\lambda\frac{\partial T}{\partial \mathrm{x}}\right] + \frac{\partial(\mu u)}{\partial \mathrm{x}}\frac{\partial u}{\partial \mathrm{x}}$

$$\left[\frac{\partial \theta}{\partial \tau} - m(\tau)\frac{\partial \theta}{\partial z} - \frac{\partial^2 \theta}{\partial z^2}\right] = \frac{(\gamma-1)}{\gamma}\frac{\theta}{\pi}\left[\frac{\partial \pi}{\partial \tau} - m(\tau)\frac{\partial \pi}{\partial z}\right] + (\gamma-1)\varepsilon^2\left(\frac{\partial \mathrm{v}}{\partial z}\right)^2$$

$$\boxed{\left[\frac{\partial \theta}{\partial \tau} - m(\tau)\frac{\partial \theta}{\partial z} - \frac{\partial^2 \theta}{\partial z^2}\right] = \varepsilon\frac{(\gamma-1)}{\gamma}\theta\frac{\partial \pi_1}{\partial \tau} + O(\varepsilon^2)}$$

Elimination of the unsteady term to leading order

mass + energy $\quad \frac{\partial \mathrm{v}}{\partial z} = \frac{\partial^2 \theta}{\partial z^2} + O(\varepsilon)$

pressure uniform inside the flame structur

ZFK linit $\quad \beta \equiv E/k_B T_b(0) \to \infty$

$$\varepsilon \ll \epsilon \ll 1, \quad (\gamma-1)\beta\varepsilon\sigma(0) = O(1), \quad \pi = 1 + \varepsilon\pi_1(\varepsilon z,\tau)$$

multiple scale

integration across the preheated zone:

$$\beta \to \infty, \quad (1-\theta_{bf}) = O(1/\beta), \quad \theta_{bf} \equiv \theta|_{z=0};$$

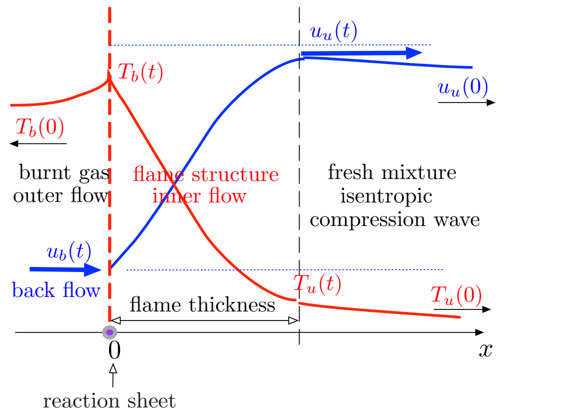
$$z = O(1): \quad \lim_{z\to\infty}\frac{\partial \theta}{\partial z} = O(1/\beta), \quad \int_{0+}^{\infty}\frac{\partial \mathrm{v}}{\partial z}\mathrm{d}z = \boxed{\mathrm{v}_u - \mathrm{v}_b = -\frac{\partial \theta}{\partial z}\Big|_{z=0^+} + O(\varepsilon)}$$

instantaneous back flow: $\quad \mathrm{v}_b(\tau) = \sigma(0)(1+\epsilon\tau)\mathrm{e}^{\beta[\theta_f(\tau)-1]/2}$

jump conditions across the reaction sheet

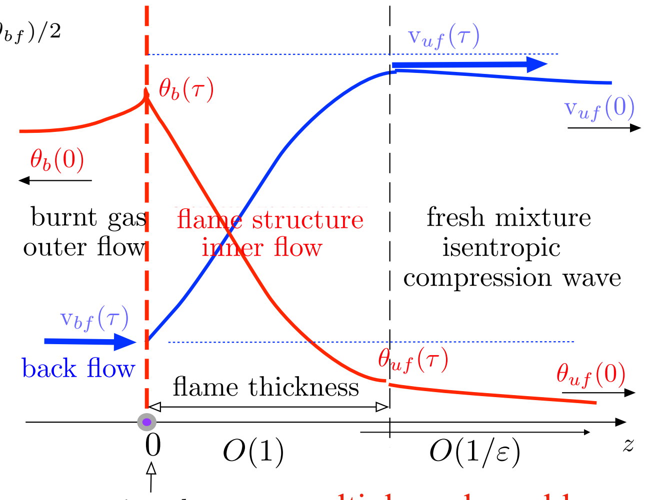
$$\boxed{-\frac{\partial \theta}{\partial z}\Big|_{z=0+} = q\mathrm{e}^{\beta(1-\theta_{bf})/2} + O(1/\beta), \quad q \equiv \frac{T_b(0)-T_u(0)}{T_b(0)} \qquad \left[\frac{\partial \theta}{\partial z} + \frac{1}{\mathrm{Le}}\frac{\partial \psi}{\partial z}\right]_{0^-}^{0^+} = O(1/\beta^2), \quad \mathrm{Le} = 1}$$

$\theta \neq (T-T_b)/(T_b-T_u)$



just ahead of the flame    back flow    flame temperature on the reaction sheet

flow velocity jump across the preheated zone: $\boxed{\mathrm{v}_{uf} - \mathrm{v}_{bf} = q\mathrm{e}^{\beta(1-\theta_{bf})/2}}$ $\qquad -\dfrac{\partial\theta}{\partial z}\Big|_{z=0^+} = q\mathrm{e}^{\beta(1-\theta_{bf})/2}$




multiple scale problem

$\pi = 1 + \varepsilon\pi_1(\varepsilon z, \tau)$ the pressure is quasi uniform inside the flame $z = O(1)$ and $\varepsilon \to 0$

$\pi_{1f}(\tau) \equiv \pi_1|_{\varepsilon z=0}$ $\quad 1 + \varepsilon\pi_{1f}(\tau) =$ reduced pressure on the flame front

matching the inner flame structure with the compression wave ahead of the flame + Riemann:

**Unsteady** compression wave ahead of the flame :
Riemann solution

$$\boxed{\begin{array}{l}\mathrm{v}_{uf}(\tau) = \mathrm{v}_{uf}(0) + \dfrac{\sqrt{1-q}}{\gamma}\pi_{1f}(\tau), \quad \mathrm{v}_{uf}(0) = \sigma(0) + q \\ \theta_{uf}(\tau) = (1-q)\left[1 + \varepsilon\dfrac{\gamma-1}{\gamma}\pi_{1f}(\tau)\right]\end{array}}$$

**Instantaneous** back flow: $\mathrm{v}_{bf}(\tau) = \sigma(\tau)m(\tau) = \sigma(0)[1+\epsilon\tau]\mathrm{e}^{\beta[\theta_{bf}(\tau)-1]/2}$

$$\sigma(0) + q + \frac{\sqrt{1-q}}{\gamma}\pi_{1f}(\tau) - \sigma(0)[1+\epsilon\tau]\mathrm{e}^{\beta[\theta_{bf}(\tau)-1]/2} = q\mathrm{e}^{\beta[\theta_{bf}(\tau)-1]/2}$$

$\pi_{1f}(\tau)$ in terms of $\theta_{bf}(\tau)$ and $\tau$

Solution of the unsteady inner structure of the flame $\Rightarrow \theta_{bf}(\tau)$ *as a functional of* $\pi_{1f}(\tau) \Rightarrow$ equation for $\pi_{1f}(\tau)$

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Instantaneous back flow+Inner structure of the flame **and** burned gas flow in quasi-steady state

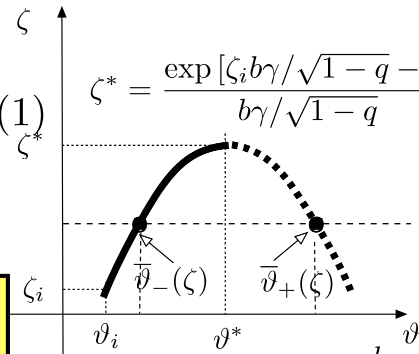
$$\overline{\theta}_{bf}(\tau) = \theta_{uf} + q, \qquad \overline{\theta}_{bf} - 1 = \varepsilon(1-q)\frac{\gamma-1}{\gamma}\pi_{1f} + O(\varepsilon^2), \qquad m = \mathrm{e}^{\beta[\overline{\theta}_{bf}-1]/2}$$

Similar transcendental algebraic equation for $\pi_{1f}$ as when the flame considered as a discontinuity

$$\boxed{\sigma(0) + q + (\sqrt{1-q}/\gamma)\pi_{1f} = [\sigma(0)(1+\epsilon\tau) + q]\,\exp(b\,\pi_{1f})}$$

$$b \equiv (\beta\varepsilon/2)(1-q)(\gamma-1)/\gamma = O(1)$$

$$\zeta^* = \frac{\exp\left[\zeta_i b\gamma/\sqrt{1-q} - 1\right]}{b\gamma/\sqrt{1-q}} \geqslant \zeta_i$$




| $\zeta \equiv \sigma(0)(1+\epsilon\tau) + q$ | $\vartheta \equiv b\pi_{1f}(\tau)$ |
|---|---|
| reduced elongation | reduced pressure |

$\tau = 0: \quad \zeta = \zeta_i \equiv \sigma(0) + q, \quad \vartheta = \vartheta_i = 0$
initial condition

$\beta \approx 8, \quad \varepsilon \approx 2.3 \times 10^{-2}, q \approx 0.9 \Rightarrow \dfrac{b\gamma}{\sqrt{1-q}} \approx 10^{-2}$

*energetic mixtures*

$\vartheta(\zeta)$ ? $\quad \boxed{\left[\zeta_i + \dfrac{\sqrt{1-q}}{\gamma b}\vartheta\right]\mathrm{e}^{-\vartheta} = \zeta}$

$\vartheta^* = 1 - \dfrac{b\gamma}{\sqrt{1-q}}\zeta_i$

$\vartheta^* \approx 1$

Near the turning point: $\dfrac{(\zeta - \zeta^*)}{\zeta^*} = -\dfrac{(\vartheta^* - \vartheta)^2}{2}$

$\zeta < \zeta^*$ two roots : $\begin{array}{l}\vartheta = \overline{\vartheta}_-(\zeta) < \vartheta^* \\ \vartheta = \overline{\vartheta}_+(\zeta) > \vartheta^* \quad \text{nonphysical}\end{array}$

$\mathrm{v}_{uf} = \sigma(0) + q + \left(\sqrt{1-q}/\gamma\right)\pi_{1f}(\tau)$ just ahead of the flame

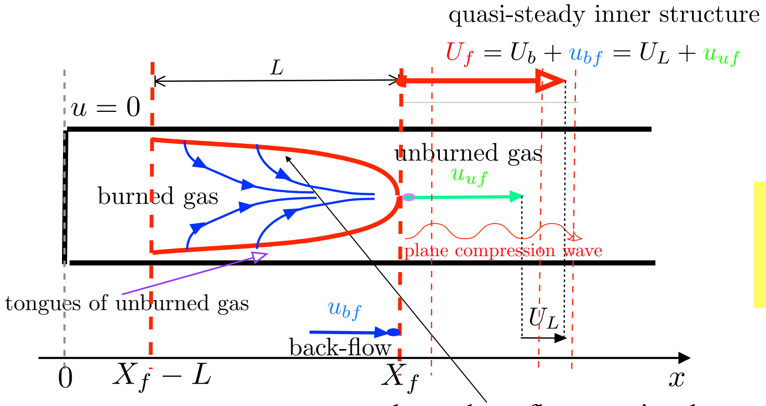
back flow

flame temperature on the reaction sheet

flow velocity jump across the preheated zone: $\mathrm{v}_{uf} - \mathrm{v}_{bf} = q\mathrm{e}^{\beta(1-\theta_{bf})/2}$

$\mathrm{v}_{bf}(t) \approx \sigma(\tau) m(\tau)$

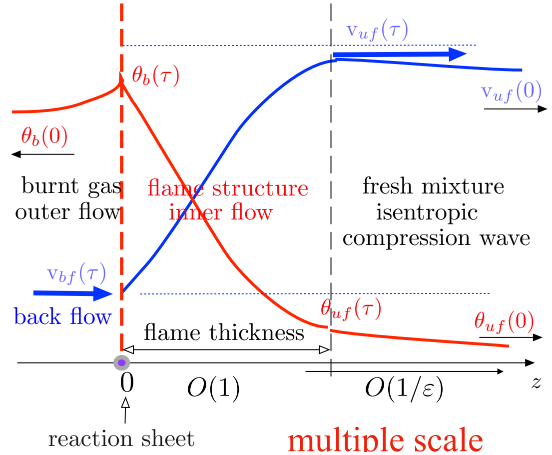
$$-\left.\frac{\partial\theta}{\partial z}\right|_{z=0^+} = q\mathrm{e}^{\beta(1-\theta_{bf})/2}$$




# Delayed back flow model

unsteady effects of the burned gas flow on the tube axis

$u_b(x,t)$




multiple scale

burned gas flow quasi-orthogonal to the wall

$$\frac{\partial u}{\partial x} \approx \frac{2}{R} U_{bw}(x,t)$$

equivalent volume source on the tube axis

$$u_{bf}(t) \equiv u_b(x = \mathrm{X}_f(t), t) = \frac{2}{R}\int_{\mathrm{X}_f - L}^{\mathrm{X}_f} U_{bw}\big(x, t - \Delta t(X_f - x)\big)\mathrm{d}x$$

delay of transmission $x \to X_f$

$$\int_{X_f - x}^{X_f} \Delta t(X_f - x)\,\mathrm{d}x = L^2/2a_b \quad \Leftarrow \quad \Delta t(X_f - x) \approx (X_f - x)/a_b$$

$$\boxed{\text{slow evolution:} \quad u_{bf}(t) \approx \frac{2L(t)}{R}\left[U_b(t) - \Delta t_w \frac{\mathrm{d}U_b}{\mathrm{d}t}\right],}$$

$\Delta t_w \frac{1}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}t} \ll 1$ Taylor expansion

$$\Delta t_w \equiv \frac{L}{2a} \qquad \Delta\tau_w \equiv \frac{\Delta t_w}{\tau_L(0)} = \frac{\Delta t_w}{d_L(0)/U_b(0)} \equiv \varepsilon\frac{L}{2d_L(0)}$$

$$\mathrm{v}_{bf} \approx \sigma(\tau)\left[m(\tau) - \Delta\tau_w \frac{\mathrm{d}m}{\mathrm{d}\tau}\right], \qquad m(\tau) = \mathrm{e}^{b\pi_{1f}(\tau)} \Leftarrow$$

inner structure of the flame in steady state

$m = \mathrm{e}^{\beta[\overline{\theta}_{bf}-1]/2}$ $\quad \beta[\overline{\theta}_{bf} - 1]/2 = b\pi_{1f}$

$\sigma(\tau) = \sigma(0)[1 + \epsilon\,\tau]$ $\quad$ ($\mathrm{v}_{bf} \neq \sigma(\tau)m(\tau)$ instantaneous back flow)

$$\mathrm{v}_{bf} \approx \sigma(0)\,[1+\epsilon\tau]\,\mathrm{e}^{b\pi_{1f}}\left[1 - \Delta\tau_w\, b\frac{\mathrm{d}\pi_{1f}}{\mathrm{d}\tau}\right],$$

$$\pi_{1f}(\tau_1), \quad \tau_1 \equiv \epsilon\tau \Rightarrow \quad \mathrm{v}_{bf} \approx \sigma(0)\mathrm{e}^{b\pi_{1f}}\left[(1+\tau_1) - \epsilon\Delta\tau_w\, b\frac{\mathrm{d}\pi_{1f}}{\mathrm{d}\tau_1}\right]$$

valid up to order $\epsilon$ $\qquad \varepsilon \ll \epsilon \ll 1$

$\Delta\tau_w = O(1)$

Non autonomous differential equation (nonlinear) of first order for the pressure on the flame $\pi_{1f}(\tau_1)$

$$\sigma(0) + q + \frac{\sqrt{1-q}}{\gamma}\pi_{1f} - \mathrm{e}^{b\pi_{1f}}\left[q + \sigma(0)(1+\tau_1) - \boxed{\epsilon\sigma(0)\Delta\tau_w\, b\frac{\mathrm{d}\pi_{1f}}{\mathrm{d}\tau_1}}\right] = 0$$

**Unsteady effects of the burned gas flow on the tube axis keeping the inner structure of the flame in steady state**

Non autonomous differential equation (nonlinear) of first order for the pressure on the flame $\pi_{1f}(\tau_1)$

$$\sigma(0)+q+\frac{\sqrt{1-q}}{\gamma}\pi_{1f}-\mathrm{e}^{b\pi_{1f}}\left[q+\sigma(0)(1+\tau_1)-\epsilon\sigma(0)\Delta\tau_w\, b\frac{\mathrm{d}\pi_{1f}}{\mathrm{d}\tau_1}\right]=0$$

$\tau_1\equiv\epsilon\tau$

variables of order unity

$\zeta\equiv\sigma(0)(1+\tau_1)+q$ — elongation $\Leftrightarrow$ time

$\vartheta\equiv b\pi_{1f}(\tau_1)$ — pressure on the flame

$\tau=0:\quad \zeta=\zeta_i\equiv\sigma(0)+q$ — initial condition

$\vartheta(\zeta)$ ?

$$\zeta-\left[\zeta_i+\frac{\sqrt{1-q}}{\gamma b}\vartheta\right]\exp\left(-\vartheta\right)=\left[\epsilon\sigma(0)^2\Delta\tau_w\right]\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta}$$

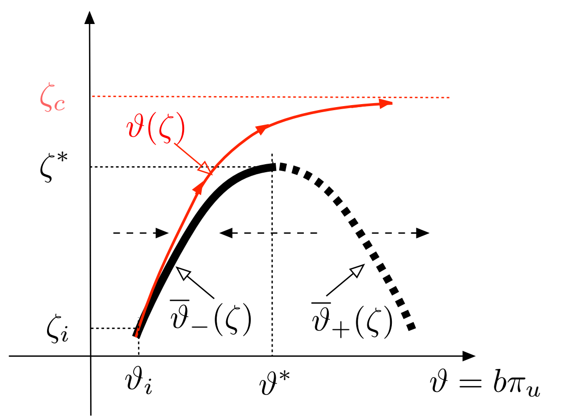
steady state eq. $\left[\zeta_i+\frac{\sqrt{1-q}}{\gamma b}\vartheta\right]\mathrm{e}^{-\vartheta}=\zeta$ — small unsteady term

Dynamics near near the turning point (saddle node bifurcation):

*see details in the next slide*

$$\frac{(\zeta-\zeta^*)}{\zeta^*}+\frac{(\vartheta^*-\vartheta)^2}{2}=\tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta},\qquad \text{where}\quad \tilde{\epsilon}_w\equiv\epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

steady state eq. — unsteady effect




**Typical equation of the catastrophe theory ! The unsteady solution diverges at finite time just above the critical time**

Generic equation of the dynamical saddle-node bifurcation

$$\frac{\mathrm{dy(t)}}{\mathrm{dt}}=\mathrm{t}+\mathrm{y}^2\qquad \lim_{\mathrm{t}\to\mathrm{t}_c}\mathrm{y(t)}=\frac{1}{\mathrm{t}_c-\mathrm{t}}-\frac{\mathrm{t}_c}{3}(\mathrm{t}_c-\mathrm{t})+\ldots\quad \underset{|\mathrm{y}(0)|\ll 1}{\text{where}}\quad \mathrm{t}_c\approx 2.338\ldots,$$

*see the reference in* Peters *et al. (2012)* Phys. Rev. E **86**, *026207*

$(1/2^{2/3})(\zeta^*/\tilde{\epsilon}_w)^{1/3}(\vartheta-\vartheta^*)\to\mathrm{y},\quad (1/2^{1/3})(\zeta^*/\tilde{\epsilon}_w)^{2/3}(\zeta-\zeta^*)/\zeta^*\to\mathrm{t}$

Small unsteady terms have a drastic effect: they make the solution diverging few time after the critical time at which the critical solution is finite in steady state ! $\lim_{\zeta\to\zeta_c}[(\vartheta-\vartheta^*)/\vartheta^*]=\infty,\quad \epsilon\ll 1:(\zeta_c-\zeta^*)/\zeta^*\propto(\tilde{\epsilon}_w/\zeta^*)^{2/3}$

**The inner structure of the flame is blown up in finite time by the divergence of the pressure; the slower the elongation growth, the close to the critical time is the blow off.**

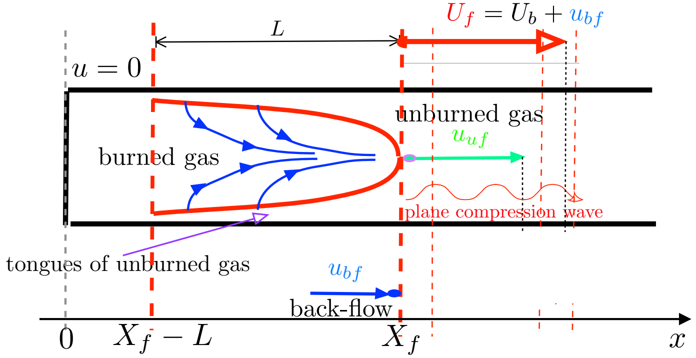
# Limitation of the asymptotic analysis

$$\frac{(\zeta-\zeta^*)}{\zeta^*}+\frac{(\vartheta^*-\vartheta)^2}{2}=\tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta},$$

$$\tilde{\epsilon}_w\equiv\epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

$$\Delta\tau_w=\frac{\Delta t_w}{t_L}=\frac{\varepsilon}{2}\frac{L}{d_L}$$

$$\tilde{\epsilon}_w=\epsilon\varepsilon\frac{\sigma(0)^2}{\zeta^*}\frac{L}{2d_L}$$




$$\boxed{\epsilon\equiv\frac{t_L(0)}{t_e},}\qquad\frac{\sigma(\tau)}{\sigma(0)}=1+\epsilon\tau,\qquad\boxed{\varepsilon\equiv\frac{U_b(0)}{a_u(0)}}$$

elongation rate | elongation | Mach number

## -1) Planar geometry near the tip ?

$$D_T/r^2\approx U_Ld_L/r^2\ll 1/t_L\approx U_L/d_L\quad\Rightarrow\quad(d_L/r)^2\ll 1$$

small modification of the flame structure by the curvature

smaller than the compressible effect $=O(\varepsilon)\,?\ \Rightarrow\ (d_L/r)^2\ll\varepsilon$

Moreover the time for an homogeneous pressure across the tube radius should be short enough

$$\epsilon\,\varepsilon\ll d_L/r\ll\sqrt{\varepsilon}\quad\Leftarrow\quad\epsilon\,U_L/a\ll d_L/r\quad\Leftarrow\quad r/a\ll t_e\equiv t_L/\epsilon$$

shorter than the evolution time

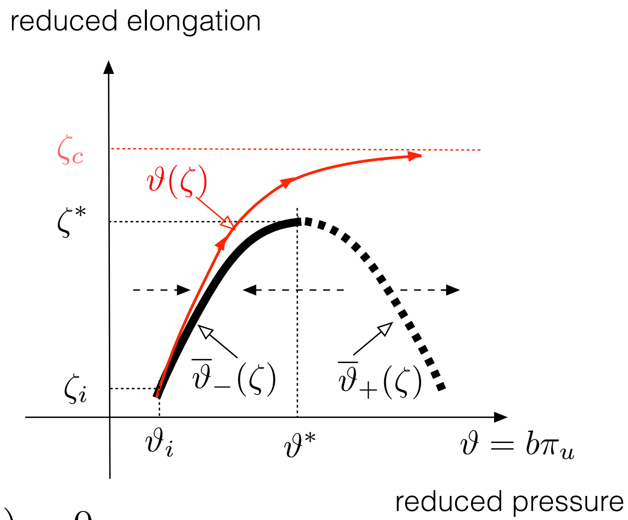
These conditions are satisfied in the distinguished limit

$$\varepsilon\ll\epsilon\ll 1,\qquad\varepsilon(r/d_L)=O(1)$$

well verified in experiments !

## Planar geometry near the tip is OK

# *Details of the calculation*




$$\zeta - \left[\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta\right]\exp\left(-\vartheta\right) = \left[\epsilon\sigma(0)^2\Delta\tau_w\right]\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta}$$

$$\zeta - Z(\vartheta) = \sigma(0)^2\Delta\tau_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta}, \qquad Z(\vartheta) \equiv \left[\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta\right]\exp\left(-\vartheta\right),$$

## *Turning point = max of* $Z(\vartheta)$

$$\left.\frac{\mathrm{d}Z}{\mathrm{d}\vartheta}\right|_{\vartheta=\vartheta^*} = 0 \qquad \frac{\sqrt{1-q}}{\gamma b}\exp\left(-\vartheta^*\right) - \left[\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta^*\right]\exp\left(-\vartheta^*\right) = 0$$

$$\frac{\sqrt{1-q}}{\gamma b}(1-\vartheta^*) = \zeta_i \qquad Z^* = \zeta^* = \left[\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta^*\right]\exp\left(-\vartheta^*\right) = \frac{\sqrt{1-q}}{\gamma b}\exp\left(-\vartheta^*\right)$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}\vartheta^2} = -\left[\frac{\sqrt{1-q}}{\gamma b} - \left(\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta\right)\right]\exp\left(-\vartheta\right) - \frac{\sqrt{1-q}}{\gamma b}\exp\left(-\vartheta\right)$$

$$\left.\frac{\mathrm{d}^2 Z}{\mathrm{d}\vartheta^2}\right|_{\vartheta=\vartheta^*} = -\frac{\sqrt{1-q}}{\gamma b}\exp\left(-\vartheta^*\right) = -\zeta^*$$

## *Expansion around the turning point* $\quad Z(\vartheta) - Z^* \approx -\frac{1}{2}\zeta^*(\vartheta-\vartheta^*)^2 + ...$

$$\zeta - \left[\zeta_i + \frac{\sqrt{1-q}}{\gamma b}\vartheta\right]\exp\left(-\vartheta\right) = \left[\epsilon\sigma(0)^2\Delta\tau_w\right]\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta} \quad \Rightarrow \quad \tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta} \approx \frac{(\zeta-\zeta^*)}{\zeta^*} + \frac{(\vartheta^*-\vartheta)^2}{2}, \quad \text{where} \quad \tilde{\epsilon}_w \equiv \epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

---

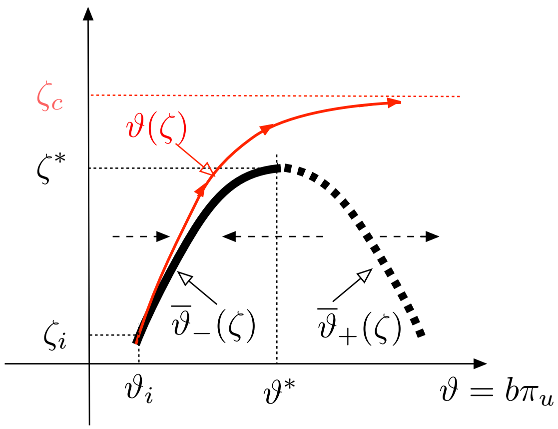
# *Unsteady inner structure of the flame*

Taking into account of the unsteadiness of the inner structure of the flame coupled to the unsteady external flows leads to results that are qualitatively similar to the previous ones provided $L/d_L > \mathrm{e}^{-2b\pi^*_{1f}}/\varepsilon \quad \Rightarrow \quad \alpha > 0$

$$\tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta} \approx \frac{(\zeta-\zeta^*)}{\zeta^*} + \frac{(\vartheta^*-\vartheta)^2}{2}, \quad \text{where} \quad \tilde{\epsilon}_w \equiv \alpha\,\epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

$$\tilde{\epsilon}_w \frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta} \approx \frac{(\zeta-\zeta^*)}{\zeta^*} + \frac{(\vartheta^*-\vartheta)^2}{2}, \quad \text{where} \quad \tilde{\epsilon}_w \equiv \alpha\,\epsilon \frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

steady state = parabola $\quad \zeta \leqslant \zeta^* : \quad \dfrac{(\vartheta^*-\overline{\vartheta})^2}{2} = \dfrac{\zeta^*-\zeta}{2}$

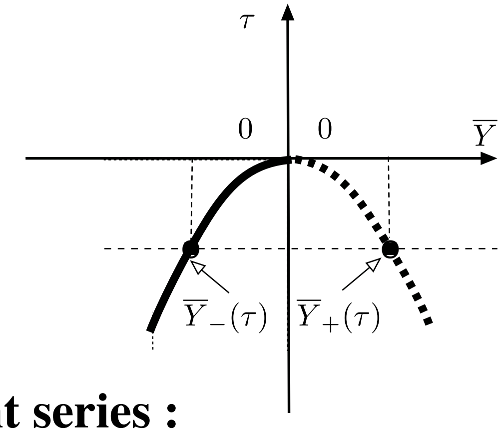



GENERIC FORM OF A DYNAMICAL **SADDLE-NODE** BIFURCATION:
**RICCATI** EQUATION

$$\frac{\mathrm{d}Y}{\mathrm{d}\tau} = \tau + Y^2$$

steady state = parabola

$\tau \leqslant 0 : \quad \overline{Y}^2 = -\tau$

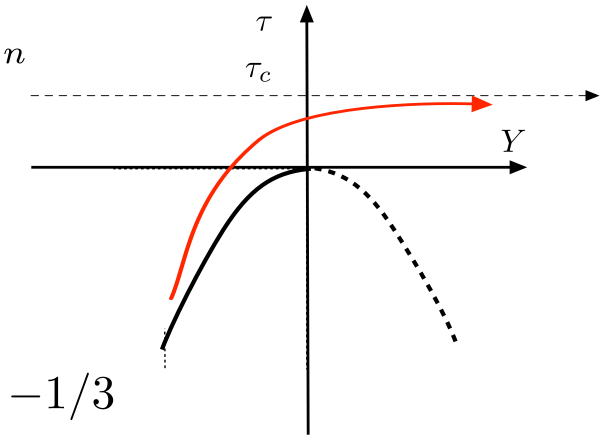



unsteady solution : finite time (spontaneous) singularity

the solution has **simple poles** around which it takes the form of a **Laurent series :**

$\tau_c > 0 : \quad \lim_{\tau-\tau_c=0^-} Y = \infty, \qquad \tau \leqslant \tau_c : \quad Y = -\dfrac{1}{\tau-\tau_c} + \sum_n a_n(\tau-\tau_c)^n$

dominant term near the singularity $\tau = \tau_c$ is solution to $\dfrac{\mathrm{d}Y}{\mathrm{d}\tau} \approx Y^2$




initial condition : $\tau \to -\infty;\ Y \to -\sqrt{\tau}$ from above then

$\tau_c = 2.238 \qquad a_1 = -1/3$

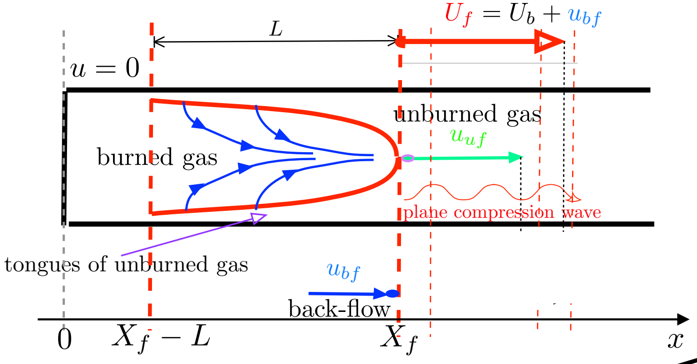
# Limitation of the asymptotic analysis

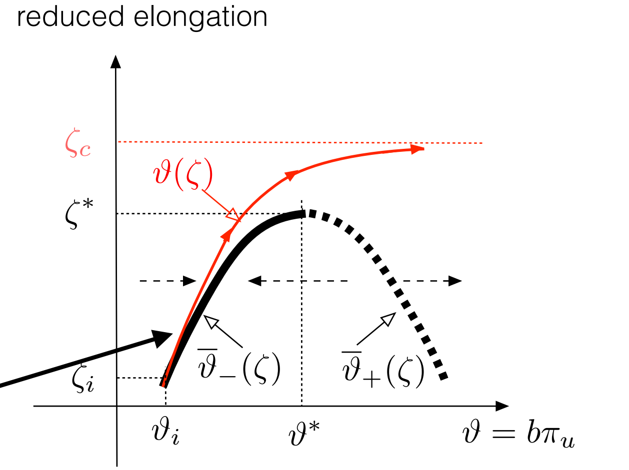
$$\frac{(\zeta-\zeta^*)}{\zeta^*}+\frac{(\vartheta^*-\vartheta)^2}{2}=\tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta},$$

$$\tilde{\epsilon}_w \equiv \epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

$$\Delta\tau_w=\frac{\Delta t_w}{t_L}=\frac{\varepsilon}{2}\frac{L}{d_L}$$

$$\tilde{\epsilon}_w=\epsilon\varepsilon\frac{\sigma(0)^2}{\zeta^*}\frac{L}{2d_L}$$




-2) Is the physical branch (black solid line) of the quasi-steady solutions stable ?

Consider the system of constitutive equations

mass+energy+ideal gas $\quad \dfrac{\partial \mathrm{v}}{\partial z}=[1-\varepsilon\pi_1]\dfrac{\partial^2\theta}{\partial z^2}-\varepsilon\dfrac{1}{\gamma}\theta\dfrac{\partial\pi_1}{\partial\tau}+O(\varepsilon^2)$

momentum $\quad \left[\dfrac{\partial \mathrm{v}}{\partial\tau}-m(\tau)\dfrac{\partial \mathrm{v}}{\partial z}-\dfrac{\partial^2\mathrm{v}}{\partial z^2}\right]=-\dfrac{1}{\gamma}\dfrac{1}{\varepsilon}\dfrac{\partial\pi_1}{\partial z}$

species $\quad \left[\dfrac{\partial\psi}{\partial\tau}-m(\tau)\dfrac{\partial\psi}{\partial z}-\dfrac{\partial^2\psi}{\partial z^2}\right]=0, \qquad \psi(z,\tau)\in[0,\ 1]$

energy $\quad \left[\dfrac{\partial\theta}{\partial\tau}-m(\tau)\dfrac{\partial\theta}{\partial z}-\dfrac{\partial^2\theta}{\partial z^2}\right]=\varepsilon\dfrac{(\gamma-1)}{\gamma}\theta\left[\dfrac{\partial\pi_1}{\partial\tau}-m(\tau)\dfrac{\partial\pi_1}{\partial z}\right], \qquad \dfrac{\partial\pi_1}{\partial z}=O(\varepsilon)$

Boundary conditions

*hyperbolic problem in the external zone* $\quad z\to\pm\infty:\ $ initial conditions $\qquad z=0:\quad \mathrm{v}=\sigma\, m(\tau) \qquad$ ZFK jump conditions for $\theta,\ \psi$

Linearize around a quasi-steady state solution for a fixed $\sigma<\sigma^*,\quad \zeta\equiv\sigma+q<\zeta^*$

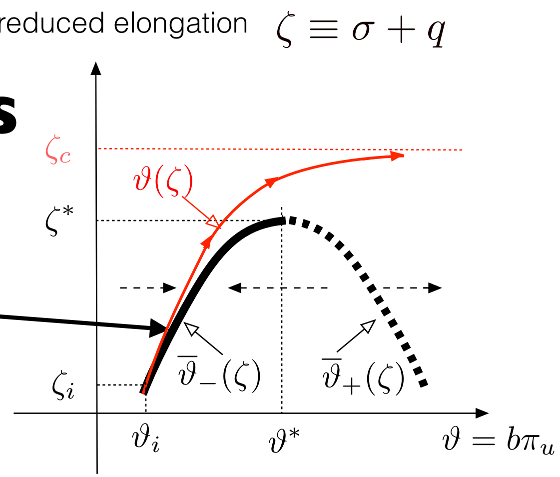
Look for solution in the form $\quad h(\xi,\tau)=\mathrm{e}^{s\tau}\tilde{h}(\xi)$. Find the complex number $s$ *(eigenvalue)*

*Clavin & al. JFM (1990)* **216,** *pp.299-322*



**If $\ \mathrm{Re}(s)>0:\ $ linear instability !**

# Limitation of the asymptotic analysis




$\pi_u \equiv \pi_1$ reduced pressure

$$b \equiv \frac{\beta\varepsilon}{2}\frac{(\gamma-1)}{\gamma}\sigma$$

## -2) Is the physical branch stable ?

***1st step***: linear response to a pressure fluctuation

*Clavin & al. JFM (1990)* **216,** *pp.299-322*

$$\left[\frac{\partial\theta}{\partial\tau} - m(\tau)\frac{\partial\theta}{\partial z} - \frac{\partial^2\theta}{\partial z^2}\right] = \varepsilon\frac{(\gamma-1)}{\gamma}\theta\left[\frac{\partial\pi_1}{\partial\tau} - \cancel{m(\tau)\frac{\partial\pi_1}{\partial z}}\right], \qquad \frac{\partial\pi_1}{\partial z} = O(\varepsilon)$$

$$\delta\theta(z,\tau) = \mathrm{e}^{s\tau}\tilde{\theta}(z,s)$$

$$\delta\theta_b(\tau) = \mathrm{e}^{s\tau}\tilde{\theta}_b(s) \quad \delta m(\tau) = \mathrm{e}^{s\tau}\tilde{m}(s)$$

*quasi-uniform pressure across the flame*

*forcing term* $s\tilde{\pi}_1$

results *(small frequency limit for simplicity)*

$$|s| \ll 1: \quad \beta\delta\theta_b(\tau) = 2(1-q)b\left[\pi_1(\tau) + \Delta\tau_\theta\frac{\partial\pi_1}{\partial\tau} + ...\right], \qquad \Delta\tau_\theta \equiv \frac{2q}{(1-q)}\frac{1}{\overline{m}^2},$$

$$\frac{\delta m(\tau)}{\overline{m}} = (1-q)b\left[\pi_1(\tau) + \Delta\tau_m\frac{\partial\pi_1}{\partial\tau} + ...\right], \qquad \Delta\tau_m = \frac{(1+q)}{(1-q)}\frac{1}{\overline{m}^2}$$

$$b \equiv \frac{\beta\varepsilon}{2}\frac{(\gamma-1)}{\gamma}\overline{\sigma} = O(1)$$

***2nd step***: **instability** mechanism for the **instantaneous** back-flow

*Clavin C.R.Acad.Sci(2023)* **351,** *pp.401-427*

$$\frac{\overline{m}}{2}(1-q)\left[\overline{\sigma}\Delta\tau_m + q\Delta\tau_\theta\right]s = \frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\overline{\pi}_1}e^{b\overline{\pi}_1}(> 0 \text{ on the physical branch}) \Rightarrow \ s > 0$$ instability !

exchange of stability at the turning point $\left.\frac{\mathrm{d}\overline{\sigma}}{\mathrm{d}\overline{\pi}_1}\right|_{\overline{\pi}_1=\overline{\pi}_1^*} = 0$

***3rd step***: **stabilisation** by the **delayed** back-flow

*Clavin JFM 2023)* ***974,*** *A46*

The **physical branch** of the quasi-steady solutions is **stable if** $\frac{L}{d_L} > \frac{2}{\varepsilon}\left[\Delta\tau_m + \frac{q}{\overline{\sigma}}\Delta\tau_\theta\right]$

this is not a very restrictive constraint; remember $r/d_L = O(1/\varepsilon)$

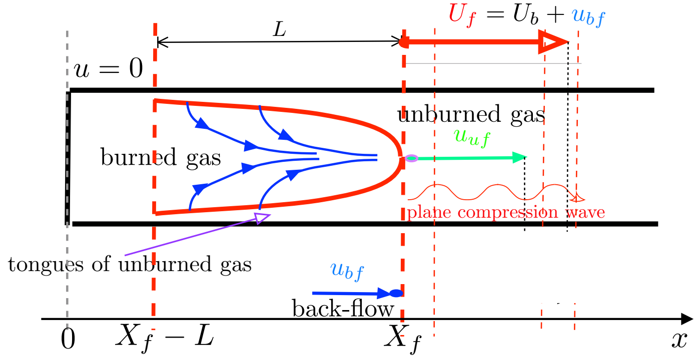
# Limitation of the asymptotic analysis

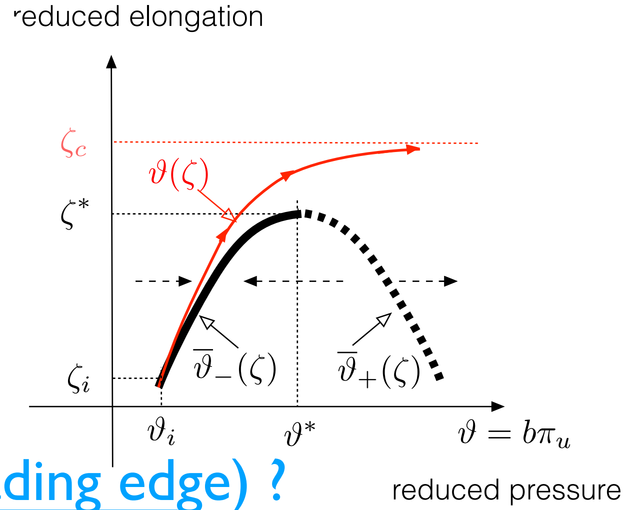
$$\frac{(\zeta-\zeta^*)}{\zeta^*}+\frac{(\vartheta^*-\vartheta)^2}{2}=\tilde{\epsilon}_w\frac{\mathrm{d}\vartheta}{\mathrm{d}\zeta},$$

$$\tilde{\epsilon}_w \equiv \epsilon\frac{\sigma(0)^2}{\zeta^*}\Delta\tau_w$$

$$\Delta\tau_w=\frac{\Delta t_w}{t_L}=\frac{\varepsilon}{2}\frac{L}{d_L}$$

$$\tilde{\epsilon}_w=\epsilon\varepsilon\frac{\sigma(0)^2}{\zeta^*}\frac{L}{2d_L}$$




## -3) **Slow dynamics** of the flame structure on the tip (leading edge) ?

Delayed back-flow: $u_{bf}(t)\approx\frac{2L}{R}\left[U_b-\Delta t_w\frac{\mathrm{d}U_b}{\mathrm{d}t}\right]$, $\boxed{\frac{\Delta t_w}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}t}\overset{?}{=}O(\epsilon)}\Rightarrow\frac{1}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}\tau}=O\left(\frac{\epsilon}{\varepsilon}\frac{d_L}{L}\right)$

$U_b\propto 1/(t_c-t)\Rightarrow$ Taylor expansion ok up to $\frac{t_c-t}{t_L}=O\left(\frac{\Delta\tau_w}{\epsilon}\right)=O\left(\frac{\varepsilon}{\epsilon}\frac{L}{d_L}\right)$ $\quad\frac{t_c-t}{t_L}\ll 1\quad\Leftrightarrow\quad 1\ll\frac{L}{d_L}\ll\frac{\epsilon}{\varepsilon}$

$(1/U_b)\mathrm{d}U_b/\mathrm{d}t=1/(t_c-t)$

$\zeta\equiv\sigma(0)(1+\epsilon\tau)+q\qquad\sigma(0)=2L/R\qquad\zeta_c-\zeta=\epsilon\sigma(0)(\tau_c-\tau)=O\left(\sigma(0)\varepsilon\frac{L}{d_L}\right)$

$\zeta_i=2L/R+q$

Expansion around the turning point $\qquad\frac{\mathrm{d}}{\mathrm{d}\tau}=\epsilon\sigma(0)\frac{\mathrm{d}}{\mathrm{d}\zeta}\qquad\sigma(0)\frac{1}{U_b}\frac{\mathrm{d}U_b}{\mathrm{d}\zeta}=O\left(\frac{1}{\varepsilon}\frac{d_L}{L}\right)$

$(1/2^{2/3})(\zeta^*/\tilde{\epsilon}_w)^{1/3}(\vartheta-\vartheta^*)\to\mathrm{y},\quad(1/2^{1/3})(\zeta^*/\tilde{\epsilon}_w)^{2/3}(\zeta-\zeta^*)/\zeta^*\to\mathrm{t}\qquad\lim_{\zeta\to\zeta_c}[(\vartheta-\vartheta^*)/\vartheta^*]=\infty,\quad\epsilon\ll 1:(\zeta_c-\zeta^*)/\zeta^*\propto(\tilde{\epsilon}_w/\zeta^*)^{2/3}$

$$\frac{\mathrm{dy(t)}}{\mathrm{dt}}=\mathrm{t}+\mathrm{y}^2$$

$$\lim_{\substack{\mathrm{t}\to\mathrm{t_c}\\|\mathrm{y}(0)|\ll 1}}\mathrm{y(t)}=\frac{1}{\mathrm{t_c-t}}-\frac{\mathrm{t_c}}{3}(\mathrm{t_c-t})+\ldots\quad\text{where}\quad\mathrm{t_c}\approx 2.338\ldots,$$

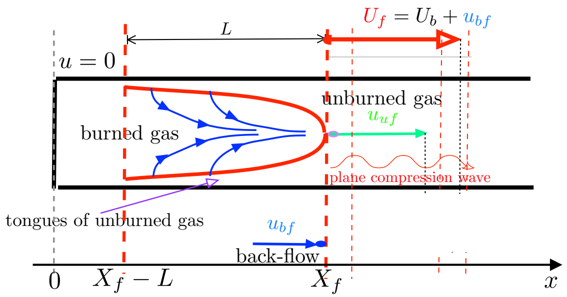
$$\lim_{\zeta\to\zeta_c}(\vartheta-\vartheta^*)=2\frac{\tilde{\epsilon}_w}{\zeta^*}\frac{\zeta^*}{\zeta_c-\zeta}\boxed{=O\left(\epsilon\frac{\sigma(0)}{\zeta^*}\right)}$$

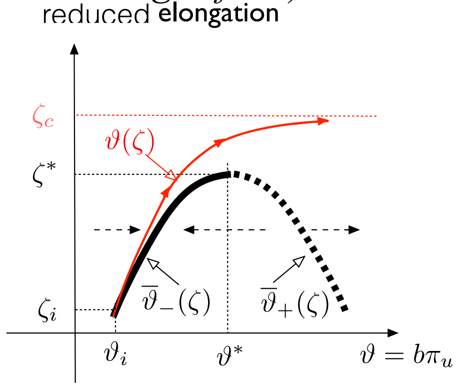
Validity of the assumption of slow evolution for $\vartheta>\vartheta^*$ : $\quad(\vartheta-\vartheta^*)<\epsilon$

The assumption is not valid in the final stage of the runaway: strong flame acceleration just above the critical velocity ! However a **finite time singularity is not doubtful.**

# Conclusion of the back-flow induced DDT

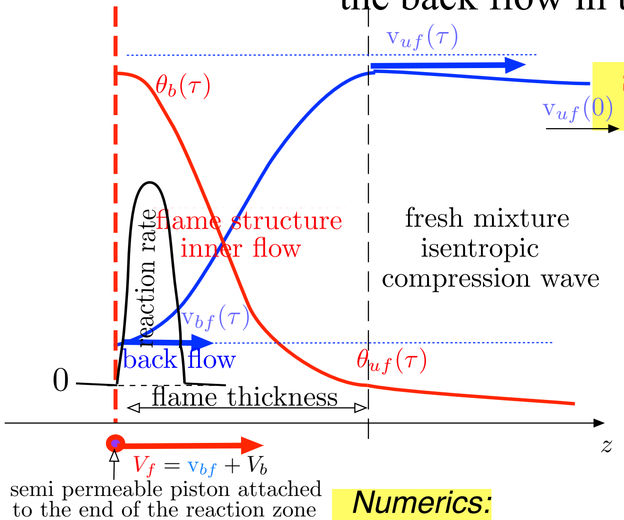
In the back-flow induced DDT scenario of an elongated flame increasing slowly, the transition to detonation occurs on the tip few time after the critical length is reached ( *the latter corresponding to the quasi-steady solution neglecting the unsteadiness processes inside the inner flame structure and also in the burned gas flow)*.




In this scenario, the sudden pressure and temperature run-away corresponds to a catastrophic process described by the finite time singularity of the approximated solution of the equations controlling the flame structure coupled to the compressible external flows. In terms of the catastrophe theory, the transition corresponds to a dynamical saddle-node bifurcation.

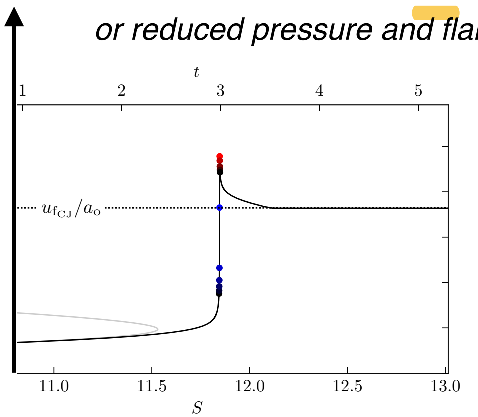
The physical mechanism is a nonlinear thermal loop between the downstream running compression waves in the external flow of unburned gas and the flame acting like a semi-transparent piston whose velocity increases linearly with the time with a coefficient of proportionality which is a strongly increasing function of the gas temperature, delayed by the unsteadiness of the back flow in the burned gas issued from the lateral flames of the elongated front.




semi permeable piston attached to the end of the reaction zone $V_b$

$$V_f = \mathrm{v}_{bf} + V_b$$
$$\mathrm{v}_{bf} = \sigma(t) V_b$$

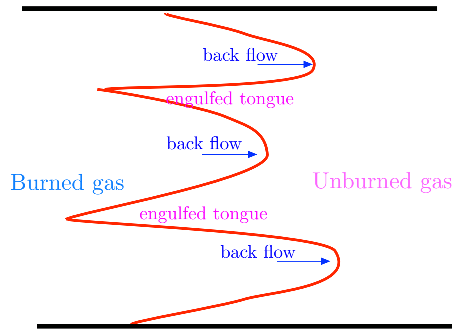
$V_b$ *laminar flame velocity taken at the CJ condition*



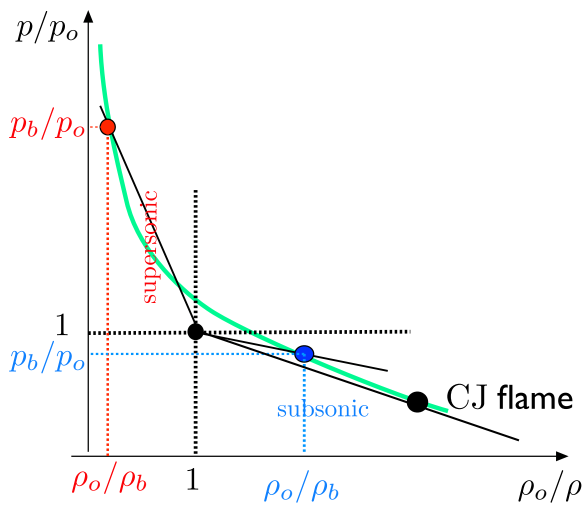

*Numerics:*
R. Hernandez-Sanchez., B. Denet (2024) *Combust. Flame,* **270** 113775

The mechanism of back-flow induced DDT can be extended to unconfined cellular flames and/or to turbulent flames in the wrinkled regime.

The velocity of a laminar flame element can approach the **sound speed in the laboratory** frame only if it is **convected by a fast flow**, the flame velocity relative to the flow being always markedly subsonic ! The flame on the tip is convected by the back flow which is proportional to the laminar flame speed times a factor between 5 and 10, resulting from the increase in surface area of the wrinkled lame front

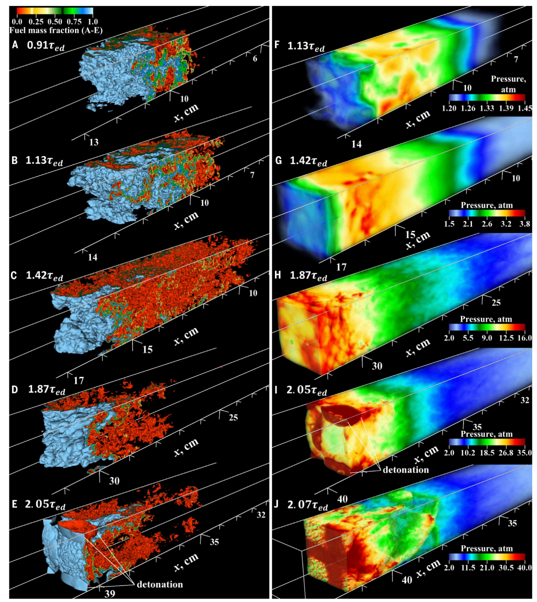



This back-flow induced DDT mechanism has **nothing to do** with the unstable **CJ flame regime** mentioned in the past, see the book of J.H.S. Lee, Cambridge (2008).

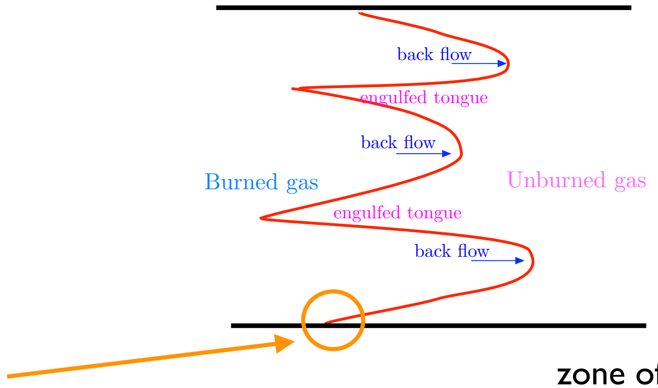


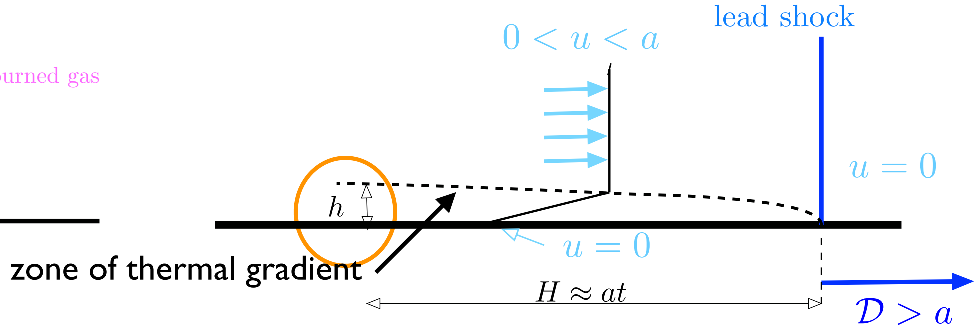

Numerical simulation of turbulent flame

Hytovick et al. (2023) Phys. Fluids 35 046112
Poludnenko et al. (2019) *Science* **366,** 7365
Poldunenko et al. (2011) *PRL* **107**, 054501



# Multi-dimensional mechanisms of transition




## DDT initiation in the boundary ?

Urtiew & Oppenheim (**1966**), Ballossier et al. **(2023),** Dexter-Brown & Jayachandran **(2024)**

Growth rate of the boundary layer ( *Dimensional analysis)*

$$h \approx \sqrt{Dt} \approx \sqrt{DH/a}\,(\approx \sqrt{lH} \approx \text{ few mms, } l \approx \text{ mean free path, } D \approx la\,)$$

$$D/h^2 \approx a/H$$

$$\frac{1}{h}\frac{\partial h}{\partial t} \approx \frac{1}{2t} \quad \Rightarrow \quad \text{growth rate: } \frac{1}{t_{gr}} \approx \frac{a}{H}$$

viscous dissipation: $c_v \dfrac{\mathrm{D}T}{\mathrm{D}t} \approx D\left(\dfrac{\partial u}{\partial z}\right)^2 \approx \dfrac{D}{h^2}u^2 \quad \Rightarrow \quad \dfrac{\Delta T}{T}\dfrac{1}{t_{gr}} \approx \dfrac{a}{H}\dfrac{u^2}{c_v T} \quad \Rightarrow \quad \dfrac{\Delta T}{T} \approx \dfrac{u^2}{c_v T} \approx \gamma(\gamma-1)\left(\dfrac{u}{a}\right)^2$

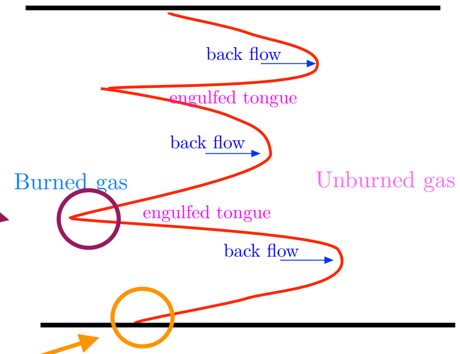
$\Delta T$ quasi uniform $\quad u/a = O(1) \Rightarrow \Delta T/T_u \approx 1/2$

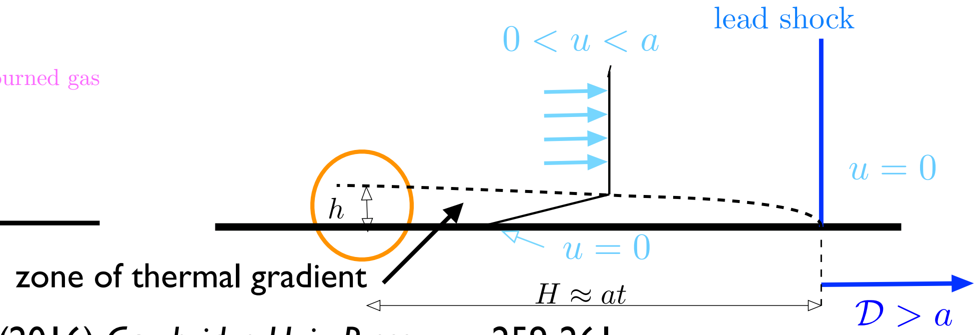
**Dissipation in the boundary layer promotes an earlier transition to detonation by an increase in temperature larger than in the bulk**

Also, the Zel'dovich's gradient mechanism cannot be excluded in the boundary layer

# Can multi-dimensional mechanisms of DTT exist ?

**On the trailing edge ?**




**In the boundary layer ?**

*Dimensional analysis:* P. Clavin, G. Searby (2016) *Cambridge Univ. Press* p.p. 259-261

*Numerics in submillimetre channels:* R.W. Houin et al. (2016) *Combust Theor. Model.* **20** (6), 1068-1087

$h \approx \sqrt{Dt} \approx \sqrt{DH/a} \approx \sqrt{lH} \approx$ few mms ( $l \approx$ mean free path, $D \approx la$ )

Dissipation induced thermal gradient coupled to the compressional heating

viscous dissipation $\quad \dfrac{\partial T/\partial t}{T} \approx D\dfrac{|\nabla T|^2}{T^2}, \quad \dfrac{1}{T}\dfrac{\partial T}{\partial t} \approx \dfrac{a}{H}\dfrac{\Delta T}{T}, \quad D\dfrac{|\nabla T|^2}{T^2} \approx \dfrac{D}{h^2}\dfrac{\Delta T^2}{T^2} \approx \dfrac{a}{H}\dfrac{\Delta T^2}{T^2} \Rightarrow \dfrac{\Delta T}{T} = O(1)$

+ compressional heating $\quad \Rightarrow T > T_c :$ **The Zel'dovich's gradient mechanism is possible**

---

**On the trailing edge ?** No back-flow!..Collision of two flame fronts with a sharp angle $\alpha \ll 1$

*Numerics:* S.Taileb et al. (2023). *Cambridge Physics of Fluids* DOI: 10.1063/5.0156876

The longitudinal velocity of the apex $U_a = U_L/\tan\alpha$ is large and can approach the speed of sound $a$

$\Rightarrow$ Intermittency of the flame surface area. $\quad U_a \approx a \Rightarrow$ heat release in phase with the pressure

**Thermo-acoustic induced DDT ?**

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More works remain to be performed on these topics

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

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ECM & CNRS

# Lecture XII
# Galloping detonations

# Lecture 12 : **Galloping detonations**

## 12-1. Physical mechanisms

*Instability mechanism*
*Two limiting cases*

## 12-2. General formulation

*Constitutive equations*
*Strong shock in the Newtonian approximation*

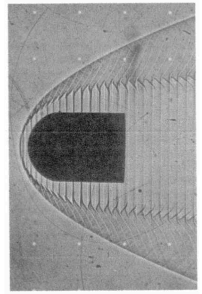
## 12-3. Strongly overdriven regimes in the limit $(\gamma - 1) \ll 1$

*Distinguished limit*
*Integral-differential equation for the dynamics*
*Oscillatory instability*

## 12-4. CJ detonations for small heat release

*Reactive Euler equations in 1-D geometry*
*Near CJ regimes for small heat release. Transonic reacting flows*
*Slow time scale*
*Asymptotic model for CJ or near CJ regimes*
*Results for simplified chemical kinetics*

# XII-1) One-dimensional physical mechanisms

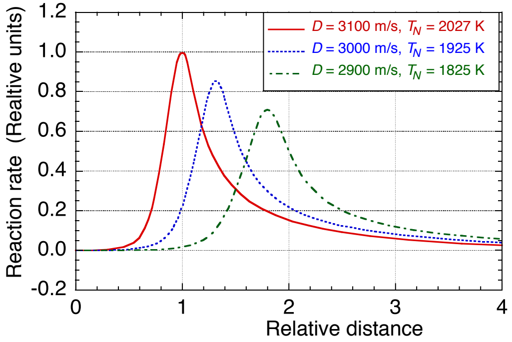


Lehr 1972

Detonation = inner shock followed by an exothermal reaction zone

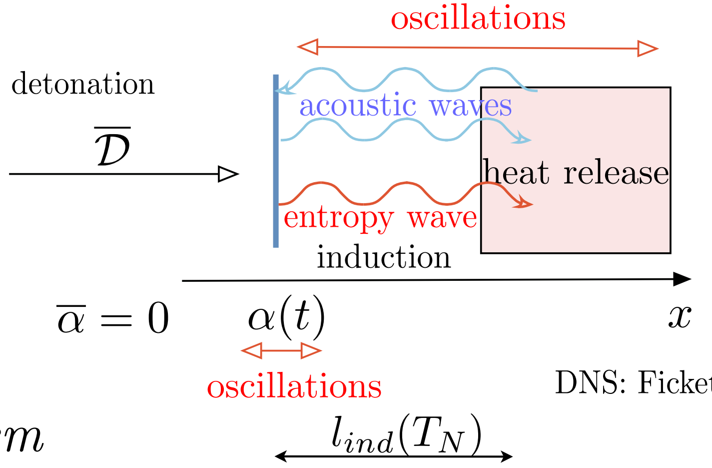
Inner structure = uniform induction zone + zone of heat release

Galloping detonation = oscillatory instability : oscillation of the velocity of the lead shock




$l_{ind}(T_N)$

reference frame of the unperturbed detonation




DNS: Ficket Woods 1966

*Instability mechanism*

$\delta\mathcal{D} = -\dot{\alpha}_t \neq 0 \Rightarrow \delta T_N \Rightarrow \delta l_{ind}$ motion of the heat release zone produces a piston like effect

feedback loop

the unstable character depends on the phase shift

Two different coupling mechanisms: $\begin{cases} \text{acoustic waves} \\ \text{entropy wave} \end{cases}$

*Two limiting cases*

Strongly overdriven regimes for $(\gamma - 1) \ll 1$ : quasi-isobaric flow, the delay by the acoustic waves is negligible

CJ regime for $q_m/c_pT_u \ll 1$ and $(\gamma - 1) \ll 1$ : transonic flow, the entropy wave is negligible

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*Asymptotic model for CJ or near CJ regimes*

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# Reactive Euler equations in 1-D geometry

$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \mathbf{u}.\nabla$

$\frac{1}{a\rho}\times$

$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \quad \left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p,\right) \quad p = (c_p - c_v)\rho T,$

$a^2 = \gamma\frac{p}{\rho} \Rightarrow \frac{1}{a\rho} = \frac{a}{\gamma p}$

1-$D$ : $\mathrm{D}^{\pm}/\mathrm{D}t \equiv \partial/\partial t \pm (a \pm u)\partial/\partial x$

$$\Rightarrow \frac{1}{\gamma p}\frac{\mathrm{D}^{\pm}p}{\mathrm{D}t} \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}u}{\mathrm{D}t} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}$$

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}} \quad \boxed{\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\dot{\mathrm{w}}}{t_N},} \quad \boxed{\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N},} \quad \Rightarrow \pm\left(\frac{1}{\gamma p}\frac{\mathrm{D}p}{\mathrm{D}t} + \nabla.\mathbf{u} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}\right)$$

entropy equation

$\dot{\mathrm{w}}(\psi, T)$

1-D Euler (compressible) eqs.

$$\boxed{\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]p \pm \frac{1}{a}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}}$$

generalized acoustic eqs.
$(\delta p = \pm\rho a\delta u)$
(useful form for the following)

# Continuous set of feed back loops

upstream running acoustic wave

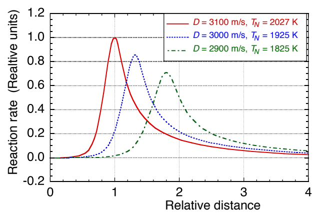
$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}\right]p - \frac{1}{a}\left[\frac{\partial}{\partial t} + (u - a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}$$

downstream running acoustic wave

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}\right]p + \frac{1}{a}\left[\frac{\partial}{\partial t} + (u + a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}$$

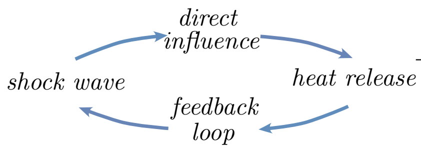
acoustic waves

entropy wave

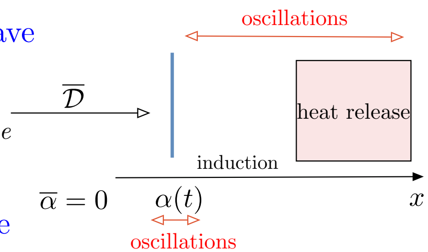


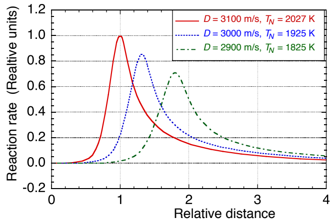

entropy wave wave

downstream running acoustic wave




upstream running acoustic wave






Galloping detonation = pulsating instability of the 1-D solution

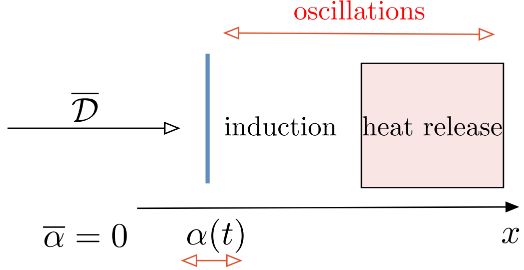
lead shock oscillates

$x = \alpha(t), \qquad \dot{\alpha}(t) \equiv \mathrm{d}\alpha/\mathrm{d}t$

velocity of the shock oscillates

reaction rate oscillates

$$\boxed{\mathcal{D}(t) = \overline{\mathcal{D}} - \dot{\alpha}_t,}$$




# Reduced equations

## *Reactive Euler equations*

$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + \mathbf{u}.\nabla$ $\qquad \rho c_p \mathrm{D}T/\mathrm{D}t = \mathrm{D}p/\mathrm{D}t + \rho q_m \dot{\mathrm{w}}$

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla.\mathbf{u}, \qquad \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -\nabla p, \qquad p = (c_p - c_v)\rho T \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\overline{t}_N}, \qquad \frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\dot{\mathrm{w}}(\psi, T)}{\overline{t}_N}$$

$$\overline{t}_N \equiv \tau_r(\overline{T}_N)$$

## *Reduced mass weighted distance from the shock* (useful for unsteady 1-D problems)

$$\mathrm{x} \equiv \frac{1}{\rho_u \overline{\mathcal{D}}\overline{t}_N}\int_{\alpha(t)}^{x} \rho(x', t)\mathrm{d}x', \qquad \mathrm{t} \equiv \frac{t}{\overline{t}_N}, \qquad \overline{t}_N \equiv \tau_r(\overline{T}_N)$$

$x = \alpha(t)$ instantaneous shock position

reaction time at the Neumann state of the unperturbed solution

$\dot{\alpha}_t \equiv \mathrm{d}\alpha/\mathrm{d}t$

$$\frac{\partial}{\partial x} = \frac{\rho}{\rho_u}\frac{1}{\overline{\mathcal{D}}\overline{t}_N}\frac{\partial}{\partial \mathrm{x}} \qquad \boxed{\overline{t}_N\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial \mathrm{t}} + \mathrm{m}(\mathrm{t})\frac{\partial}{\partial \mathrm{x}},}$$

where

$$\mathrm{m}(\mathrm{t}) \equiv \left[\frac{\rho(x,t)[u(x,t) - \dot{\alpha}_t]}{\rho_u \overline{\mathcal{D}}}\right]_{x=\alpha(t)} = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

reduced mass flux across the lead shock

$$\begin{cases} u|_{\mathrm{x}=0}(t) - \dot{\alpha}_t = u_N(t) \\ \rho_N(t)u_N(t) = \rho_u(\overline{\mathcal{D}} - \dot{\alpha}_t) \end{cases}$$

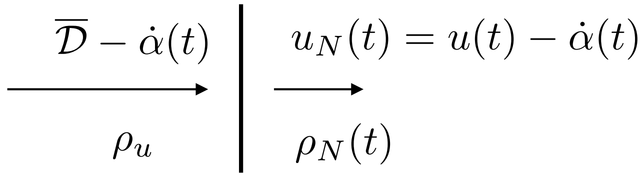
mass conservation

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial \mathrm{t}} + \mathrm{m}(\mathrm{t})\frac{\partial}{\partial \mathrm{x}} \qquad \mathrm{m}(\mathrm{t}) = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\rho_u}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{\mathcal{D}}}\right) \quad \Leftrightarrow \quad \frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\overline{\rho}_N}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{u}_N}\right),$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{u}{\overline{\mathcal{D}}}\right) = -\frac{\partial}{\partial \mathrm{x}}\left(\frac{p}{\rho_u \overline{\mathcal{D}}^2}\right), \qquad p = (c_p - c_v)\rho T,$$

$$\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_p T}\dot{\mathrm{w}}, \qquad \frac{\mathrm{D}\psi}{\mathrm{Dt}} = \dot{\mathrm{w}},$$




referential frame of the moving shock

$$\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial \mathrm{t}} + \mathrm{m(t)}\frac{\partial}{\partial \mathrm{x}} \qquad \mathrm{m(t)} = 1 - \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\rho_u}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{\mathcal{D}}}\right) \quad \Leftrightarrow \quad \frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{\overline{\rho}_N}{\rho}\right) = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{u}_N}\right),$$

$$\frac{\mathrm{D}}{\mathrm{Dt}}\left(\frac{u}{\overline{\mathcal{D}}}\right) = -\frac{\partial}{\partial \mathrm{x}}\left(\frac{p}{\rho_u \overline{\mathcal{D}}^2}\right), \qquad p = (c_p - c_v)\rho T,$$

$$\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_p T}\dot{\mathrm{w}}, \qquad \frac{\mathrm{D}\psi}{\mathrm{Dt}} = \dot{\mathrm{w}},$$

adiabatic compression

$m(\mathrm{t})$ unknow

$\dot{\alpha}_t/\overline{\mathcal{D}} = 1 - m(t)$

## *Boundary conditions*

Neumann state $\mathrm{x} = 0: \quad \rho = \rho_N(\mathrm{t}), \quad p = p_N(\mathrm{t}), \quad T = T_N(\mathrm{t}) \quad \rho_N(t)(u - \dot{\alpha}_t) = \rho_u \overline{\mathcal{D}}\,\mathrm{m(t)} \quad \psi = 0$

expressed in terms of m(t) by the RH conditions

$$\frac{u_N}{\mathcal{D}} = \frac{\rho_u}{\rho_N} = \frac{(\gamma-1)M_u^2 + 2}{(\gamma+1)M_u^2}, \qquad \frac{p_N}{p_u} = \frac{2\gamma M_u^2 - (\gamma-1)}{(\gamma+1)},$$

$$\frac{T_N}{T_u} = \frac{\left[2\gamma M_u^2 - (\gamma-1)\right]\left[(\gamma-1)M_u^2 + 2\right]}{(\gamma+1)^2 M_u^2}, \qquad M_N^2 = \frac{(\gamma-1)M_u^2 + 2}{2\gamma M_u^2 - (\gamma-1)}.$$

$$\boxed{M_u = [1 - m(t)]\overline{M}_u}$$

Burnt gas $\mathrm{x} \to \infty$ : $\begin{cases} \text{overdriven regimes: } u = \overline{u}_b \\ \text{CJ regime: } p - \overline{p}_b = \overline{\rho}_b \overline{a}_b (u - \overline{u}_b) \end{cases}$ i.e. outgoing acoustic waves (radiation condition)

Analytical solutions are obtained in limiting cases

## Strong shock in the Newtonian approximation

$$\overline{M}_u \gg 1, \quad (\gamma-1) \ll 1 \quad \Rightarrow \quad \overline{M}_N^2 \approx \frac{\gamma-1}{2} + \frac{1}{\overline{M}_u^2} \ll 1$$

*Distinguished limit:* $\overline{M}_u^2 \gg 1, \quad (\gamma-1)M_u^2 = O(1)$

$$\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma-1) = O(\epsilon^2)$$

small parameter

$$\overline{u}_N/\overline{\mathcal{D}} = \overline{\rho}_u/\overline{\rho}_N \approx \epsilon^2, \qquad \overline{p}_N/\overline{p}_u \approx \overline{M}_u^2 = O(1/\epsilon^2), \qquad \overline{a}_N/\overline{\mathcal{D}} \approx \epsilon, \qquad (\overline{a}_N/\overline{a}_u)^2 \approx [2 + (\gamma-1)\overline{M}_u^2]/2 = O(1)$$

$$\left.\begin{array}{l} \overline{\rho u} - \overline{\rho}_N \overline{u}_N = 0 \\ \dfrac{\mathrm{d}\overline{p}}{\mathrm{d}x} + \overline{\rho u}\dfrac{\mathrm{d}\overline{u}}{\mathrm{d}x} = 0 \end{array}\right\} \quad \text{and} \quad a_N^2/\gamma = p_N/\rho_N \quad \Rightarrow \quad \boxed{(\overline{p}/\overline{p}_N - 1) = -\epsilon^2\,(\overline{u}/\overline{u}_N - 1)\,,}$$

Quasi-isobaric approximation of the shocked gas

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*Results for simplified chemical kinetics*

# Strongly overdriven detonations in the limit $(\gamma-1)\ll 1$

P. Clavin and L. He (1996) *J. Fluid Mech.*, **306**, 353-378

## Distinguished limit

$$\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma-1)=O(\epsilon^2)$$

$$\boxed{q_N \equiv q_m/c_p\overline{T}_N = O(1)} \quad \Leftrightarrow \quad \text{strongly overdriven regime}$$

$$M_{u_{CJ}} = \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q}+1} \quad \text{where } \mathcal{Q} \equiv \frac{\gamma+1}{2}\frac{q_m}{c_p T_u}$$

$$\overline{T}_N/T_u = O(1) \quad \Rightarrow M_{u_{CJ}} = O(1) \Rightarrow M_u \gg M_{u_{CJ}}$$

## Negligible adiabatic compression

$$\left.\begin{array}{r}\dfrac{1}{T}\dfrac{\mathrm{D}T}{\mathrm{Dt}} - \cancel{\dfrac{(\gamma-1)}{\gamma}\dfrac{1}{p}\dfrac{\mathrm{D}p}{\mathrm{Dt}}} = \dfrac{q_m}{c_pT}\dot{\mathrm{w}}, \\ \delta p/p = O(\epsilon^2)\end{array}\right\} \quad \Rightarrow \quad \boxed{\left\{\begin{array}{l}\dfrac{\partial T}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial T}{\partial \mathrm{x}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi,T), \\ \dfrac{\partial \psi}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial \psi}{\partial \mathrm{x}} = \dot{\mathrm{w}}(\psi,T)\end{array}\right.}$$

$$\mathrm{x}=0: \quad \psi=0, \quad T=T_N(\mathrm{t})$$

$$T_N/T_u \approx [(\gamma-1)M_u^2+2]/2 \quad \text{where } M_u^2 = \overline{M}_u^2(1-\dot{\alpha}_t/\overline{\mathcal{D}})^2$$

The solution yields T(x,t) and $\psi$(x,t) in terms of $\mathrm{m(t)} = 1-\dot{\alpha}_t/\mathcal{D}$

## *Distinguished limit*

$$\boxed{(\gamma-1)\beta_N = O(1)} \quad \Leftrightarrow \quad \boxed{\beta_N = O(1/\epsilon^2)}$$

$$\beta_N \equiv \left[\frac{T}{\dot{\mathrm{w}}}\frac{\partial \dot{\mathrm{w}}}{\partial T}\right]_{T=T_N}$$

$$m(t) = 1 - \dot{\alpha}_t/\overline{\mathcal{D}}, \quad \dot{\alpha}_t/\overline{\mathcal{D}} = O(1/\beta_N) \Rightarrow \begin{cases} \dfrac{\partial T}{\partial \mathrm{t}} + \dfrac{\partial T}{\partial \mathrm{x}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi, T), \\ \dfrac{\partial \psi}{\partial \mathrm{t}} + \dfrac{\partial \psi}{\partial \mathrm{x}} = \dot{\mathrm{w}}(\psi, T) \end{cases} \quad \mathrm{x} = 0: \ T = T_N(t), \ \psi = 0,$$

$$\frac{D}{Dt} \approx \frac{\partial}{\partial t} + \frac{\partial}{\partial \mathrm{x}}$$

$$(T_N - \overline{T}_N)/\overline{T}_N = O(1/\beta_N) \ \Rightarrow \ \boxed{\Theta_N(\mathrm{t}) \equiv \beta_N(T_N(t) - \overline{T}_N)/\overline{T}_N = O(1),} \quad \dot{\mathrm{w}} - \overline{\dot{\mathrm{w}}} = O(1)$$

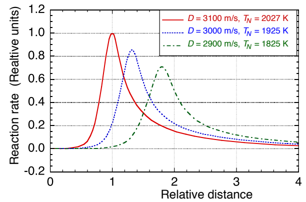
## *Steady state solutions*

$$\begin{cases} \dfrac{\mathrm{d}T}{\mathrm{dx}} = \dfrac{q_m}{c_p}\dot{\mathrm{w}}(\psi, T), \\ \dfrac{\mathrm{d}\psi}{\mathrm{dx}} = \dot{\mathrm{w}}(\psi, T) \end{cases} \qquad T_N = \ \text{cst.} \neq \overline{T}_N$$

$$\mathrm{x} = 0: \ T = T_N, \ \psi = 0,$$

$$\Theta_N \equiv \beta_N(T_N - \overline{T}_N)/\overline{T}_N$$

$$T = \mathcal{T}(\Theta_N, \mathrm{x}), \quad \psi = \mathcal{Y}(\Theta_N, \mathrm{x}), \quad \Omega(\Theta_N, \mathrm{x}) \equiv \dot{\mathrm{w}}(\mathcal{T}, \mathcal{Y})$$




## *Unsteady solution* (retarded functions)

$$\boxed{T(\mathrm{x},\mathrm{t}) = \mathcal{T}\left(\Theta_N(\mathrm{t}-\mathrm{x}), \mathrm{x}\right), \quad \psi(\mathrm{x},\mathrm{t}) = \mathcal{Y}\left(\Theta_N(\mathrm{t}-\mathrm{x}), \mathrm{x}\right), \quad \dot{\mathrm{w}} = \Omega(\Theta_N(t-\mathrm{x}), \mathrm{x})}$$

*Distinguished limit* $\boxed{(\gamma-1)\beta_N = O(1)}$ $\Leftrightarrow$ $\boxed{\beta_N = O(1/\epsilon^2)}$

$$\mathrm{m}(t) = 1 + O(1/\beta_N) \qquad \left[\frac{\partial}{\partial t} + \mathrm{m}(t)\frac{\partial}{\partial \mathrm{x}}\right] \approx \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathrm{x}}\right]$$

*Rankine-Hugoniot*

$$\frac{u_N}{\overline{u}_N} \approx 1 + \frac{\dot\alpha_t}{\overline{u}_N}$$

$$\left.\begin{aligned} \frac{T_N}{T_u} &= \frac{\left[2\gamma M_u^2 - (\gamma-1)\right]\left[(\gamma-1)M_u^2 + 2\right]}{(\gamma+1)^2 M_u^2} \\ M_u^2 &= \overline{M}_u^2(1-\dot\alpha_t/\overline{\mathcal{D}})^2 \end{aligned}\right\} \Rightarrow \quad (T_N(\mathrm{t}) - \overline{T}_N)/\overline{T}_N \approx -(\gamma-1)\dot\alpha_t/\overline{u}_N \ll 1, \quad \Rightarrow \quad \boxed{\frac{\dot\alpha_t}{\overline{u}_N} = -\frac{\Theta_N(\mathrm{t})}{\beta_N(\gamma-1)}}$$

*Quasi-isobaric approximation in the shocked gas + Continuity*

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathrm{x}}\right]\frac{T}{\overline{T}_N} \approx \left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathrm{x}}\right]\frac{\overline{\rho}_N}{\rho} = \frac{\partial}{\partial \mathrm{x}}\left(\frac{u}{\overline{u}_N}\right)$$

*Energy*

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \mathrm{x}}\right]\frac{T}{\overline{T}_N} = \frac{q_m}{c_p\overline{T}_N}\dot{\mathrm{w}} \longrightarrow \boxed{\frac{u_b}{\overline{u}_N} - \frac{u_N}{\overline{u}_N} = \frac{q_m}{c_p\overline{T}_N}\int_0^\infty \dot{\mathrm{w}}(\mathrm{x},t)\mathrm{dx}}$$

$$\dot{\mathrm{w}} = \Omega(\Theta_N(t-\mathrm{x}),\mathrm{x})$$

*Nonlinear integral-equation*

$$\boxed{1 + b\Theta_N(\mathrm{t}) = \int_0^\infty \Omega(\Theta_N(\mathrm{t}-\mathrm{x}),\mathrm{x})\mathrm{dx},} \qquad b^{-1} = \beta_N(\gamma-1)\frac{q_m}{c_p\overline{T}_N} = O(1)$$

$$\int_0^\infty \Omega(\overline{\Theta}_N,\mathrm{x})\mathrm{dx} = 1$$

(Clavin He 1996)

# strongly overdriven detonations in the Newtonian limit

## quasi-isobaric approximation in the shocked gases

+ energy eq. for $T$ heat release per unit mass progress variable

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)\left[\ln T-\cancel{\frac{(\gamma-1)}{\gamma}}\ln p\right]=\frac{q_m}{c_pT}\frac{\dot{w}(T,Y)}{t_r}$$

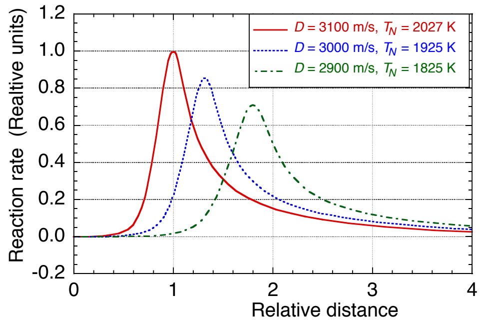
$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial r}\right)Y=\frac{\dot{w}(T,Y)}{t_r}$$

$\dot{w}$ : non-dimensional reaction rate

chemical kinetics

$$\boxed{\text{mass} \quad \frac{1}{\rho}\frac{\partial\rho}{\partial t}+\frac{\partial u}{\partial x}=0}$$

$$\boxed{\rho T\approx \text{cst.}}$$




steady state

$\Omega(\Theta_N,\mathrm{x})$

unsteady / steady distribution of the rate of heat release

$$\boxed{\Omega(\Theta_N(\mathrm{t}-\mathrm{x}),\mathrm{x})}$$

$$\Theta_N(\mathrm{t})=\frac{\beta(T_N(\mathrm{t})-\overline{T}_N)}{\overline{T}_N}$$

$$\mathrm{x}\equiv\frac{1}{t_r\rho_u\overline{\mathcal{D}}}\int_{\alpha(t)}^{x}\rho\mathrm{d}x \qquad \mathrm{t}\equiv\frac{t}{t_r}$$

$+$

$\boxed{\text{conservation of mass}}$ and boundary conditions

Rankine-Hugoniot at $\mathrm{x}=0$ : $\dfrac{\mathcal{D}(\mathrm{t})-\overline{\mathcal{D}}}{\overline{\mathcal{D}}}\Leftrightarrow\Theta_N(\mathrm{t})$ $\qquad$ $\mathrm{x}\rightarrow\infty$ : boundedness

$\Downarrow$

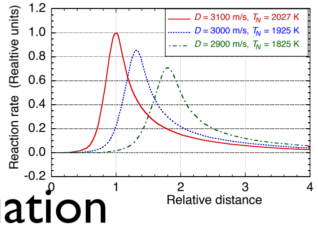
integral equation for $\Theta_N(t)$

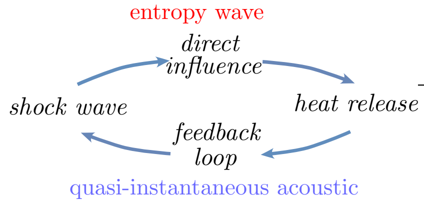
$$\boxed{1+b\Theta_N(t)=\int_0^{\infty}\Omega(\Theta_N(\mathrm{t}-\mathrm{x}),\mathrm{x})\mathrm{d}\mathrm{x},}$$

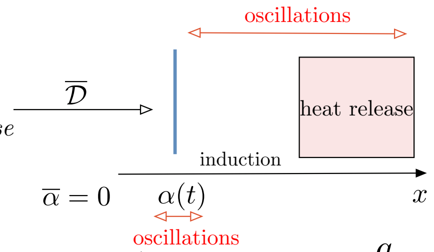
$$b^{-1}\equiv\beta_N(\gamma-1)\frac{q_m}{c_p\overline{T}_N}$$

strongly overdriven detonations in the Newtonian limit

quasi-isobaric approximation in the shocked gas




**Integral equation for the reduced temperature**

$$1 + b\Theta_N(t) = \int_0^\infty \Omega(\Theta_N(\mathrm{t}-\mathrm{x}), \mathrm{x})\mathrm{dx}, \qquad b^{-1} \equiv \beta_N(\gamma-1)\frac{q_m}{c_p\overline{T}_N}$$

**Arrhenius law (1 step)**

$$\Omega(\Theta_N, \mathrm{x}) \approx \mathrm{e}^{\Theta_N}\overline{\Omega}(\mathrm{e}^{\Theta_N}\mathrm{x}), \qquad \text{Arrhenius} \quad l/\bar{l} = \mathrm{e}^{-\Theta_N}$$

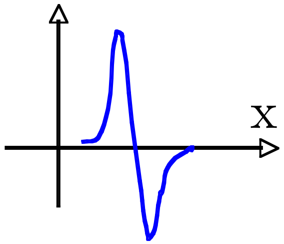
$$\boxed{\Omega(\Theta_N, \mathrm{x}) \approx \mathrm{e}^{\Theta_N}\overline{\Omega}(\mathrm{e}^{\Theta_N}\mathrm{x})} \qquad \int_0^\infty \Omega(\Theta_N, \mathrm{x})\mathrm{dx} = 1$$

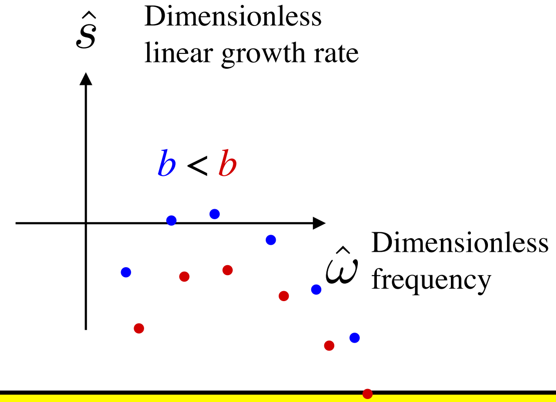
## Linearization. Normal mode analysis

$$\Theta_N(t) \propto \mathrm{e}^{\sigma t}, \qquad \sigma = \hat{s} + \mathrm{i}\hat{\omega}$$

$$\Omega'_N(\mathrm{x}) \equiv \mathrm{d}[\mathrm{x}\overline{\Omega}(\mathrm{x})]/\mathrm{dx}$$

$$\boxed{b = \int_0^\infty \Omega'_N(\mathrm{x})\mathrm{e}^{-\sigma \mathrm{x}}\mathrm{dx}}$$

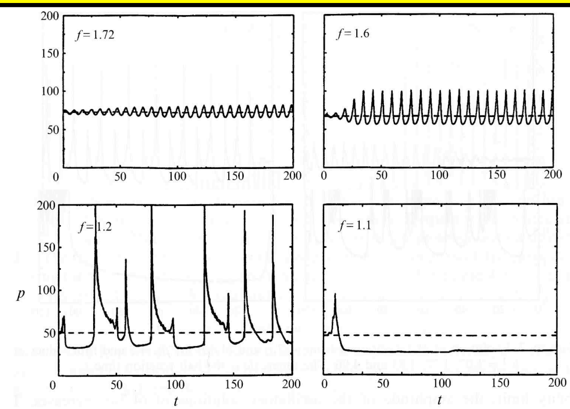



$$\hat{\omega} = O(1) \Leftrightarrow \omega = O(\bar{t}_N)$$

frequence of oscillation of order of the transit time

**Poincaré-Andronov (Hopf) bifurcation**




OK with DNS

high thermal sensitivity, $\beta_N$
and/or
stiffness of the distribution of heat release $\Omega(\Theta_N, \mathrm{x})$

promote the instability

intermittency

**nonlinear effects: stochasticity + dynamical quenching**

# Lecture 12 : **Galloping detonations**

## 12-1. Physical mechanisms

*Instability mechanism*
*Two limiting cases*

## 12-2. General formulation

*Constitutive equations*
*Strong shock in the Newtonian approximation*

## 12-3. Strongly overdriven regimes in the limit $(\gamma - 1) \ll 1$

*Distinguished limit*
*Integral-differential equation for the dynamics*
*Oscillatory instability*

## 12-4. CJ detonations for small heat release

*Reactive Euler equations in 1-D geometry*
*Near CJ regimes for small heat release. Transonic reacting flows*
*Slow time scale*
*Asymptotic model for CJ or near CJ regimes*
*Results for simplified chemical kinetics*

## Reactive Euler equations in 1-D geometry

$D/Dt \equiv \partial/\partial t + \mathbf{u}.\nabla$ $\frac{1}{a\rho}\times$ $a^2 = \gamma p/\rho$ 1-$D$ : $\mathrm{D}^{\pm}/\mathrm{Dt} \equiv \partial/\partial t \pm (a \pm u)\partial/\partial x$

$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{Dt}} = -\nabla.\mathbf{u}$, $\left(\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{Dt}} = -\nabla p,\right)$ $p = (c_p - c_v)\rho T$, $\Rightarrow \frac{1}{\gamma p}\frac{\mathrm{D}^{\pm}p}{\mathrm{Dt}} \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}u}{\mathrm{Dt}} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}$

$\boxed{\frac{\mathrm{D}}{\mathrm{Dt}} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}}$ $\boxed{\frac{\mathrm{D}\psi}{\mathrm{Dt}} = \frac{\dot{\mathrm{w}}}{t_N},}$ $\boxed{\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{Dt}} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{Dt}} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N},}$ $\Rightarrow \pm\left(\frac{1}{\gamma p}\frac{\mathrm{D}p}{\mathrm{Dt}} + \nabla.\mathbf{u} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}\right)$

entropy equation

$\dot{\mathrm{w}}(\psi, T)$

1-D Euler (compressible) eqs.

$$\boxed{\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u\pm a)\frac{\partial}{\partial x}\right]p \pm \frac{1}{a}\left[\frac{\partial}{\partial t} + (u\pm a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{t_N}}$$

generalized acoustic eqs.
$(\delta p = \pm\rho a\delta u)$
(useful form for the following)

# CJ detonations for small heat release

P. Clavin and F.A. Williams (2002) *Combust. Theor. Model.*, **6**, 127-129

## Near CJ regimes for small heat release. Transonic reacting flows

*CJ and overdriven regimes*

$$M_{u_{CJ}} = \sqrt{\mathcal{Q}} + \sqrt{\mathcal{Q}+1} \qquad \mathcal{Q} \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_pT_u} \qquad f \equiv \frac{\left(\overline{M}_u - \overline{M}_u^{-1}\right)^2}{4\mathcal{Q}}$$

CJ regime: $f = 1$, overdriven regime: $f > 1$

*Small heat release approximation* (transonic regimes)

$$\mathcal{Q} \ll 1$$

small parameter: $\boxed{\epsilon^2 \equiv \mathcal{Q} \ll 1}$ $\boxed{M_{u_{CJ}}^2 \approx 1 + 2\epsilon}$ overdriven regime near CJ: $\boxed{f = O(1)}$

$\overline{M}_u^2 \approx 1 + 2\epsilon\sqrt{f}$ $\overline{M}_u^2 - M_{u_{CJ}}^2 \approx 2\epsilon(\sqrt{f} - 1)$

⚠ $\epsilon$ in p.9 $\neq$ $\epsilon$ in p.5

$$q_N \equiv q_m/c_pT_N = O(\epsilon^2) \qquad f = O(1)$$

*Rankine-Hugoniot conditions* $\quad (M_u^2 - 1) \ll 1$

see p.6 lecture X

$$\frac{T_N}{T_u} \approx 1 + (\gamma-1)(M_u^2-1), \qquad \frac{p_N}{p_u} \approx \frac{\rho_N}{\rho_u} \approx 1 + (M_u^2 - 1), \qquad \psi = 0$$

*Steady state* $\quad (\overline{M}_u^2 - 1) \approx (1 - \overline{M}_N^2) \approx 2\epsilon\sqrt{f}$

heat release

compressible effect

$$\frac{\overline{T} - \overline{T}_N}{T_u} \approx \epsilon^2\psi - (\gamma-1)\epsilon\sqrt{f}[1 - \sqrt{1-(\psi/f)}], \qquad (1 - \overline{M}_b^2) \approx 2\epsilon\sqrt{f-1}$$

$$\frac{(\gamma+1)}{2\gamma}\frac{(\overline{p}-p_u)}{p_u} = \frac{(\gamma+1)}{2}\overline{M}_u^2\frac{(\overline{\mathcal{D}}-\overline{u})}{\overline{\mathcal{D}}} \approx \epsilon\sqrt{f}\overline{M}_u[1 + \sqrt{1-(\psi/f)}],$$

crossover temperature : $\quad T_u < T^* < T_N$ $\qquad$ separation of scale: $\tau_{coll}/\overline{t}_N \ll (\overline{M}_u - 1) \Rightarrow \mathrm{e}^{-E/k_BT_N} \ll \epsilon$

*Distinguished limit* $\quad \boxed{(\gamma - 1) = O(\epsilon)} \Rightarrow (T - T_N)/T_N = O(\epsilon^2)$

*Non-dimensional equations* $\quad \mathrm{t} \equiv \frac{t}{\overline{t}_N}, \quad \mathrm{x} \equiv \frac{x}{a_u\overline{t}_N}, \quad \breve{u} \equiv \frac{u}{a_u}, \quad \breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{p}{p_u}\right), \quad \breve{\theta} \equiv \frac{(T - \hat{T}_u)}{\hat{T}_u}$

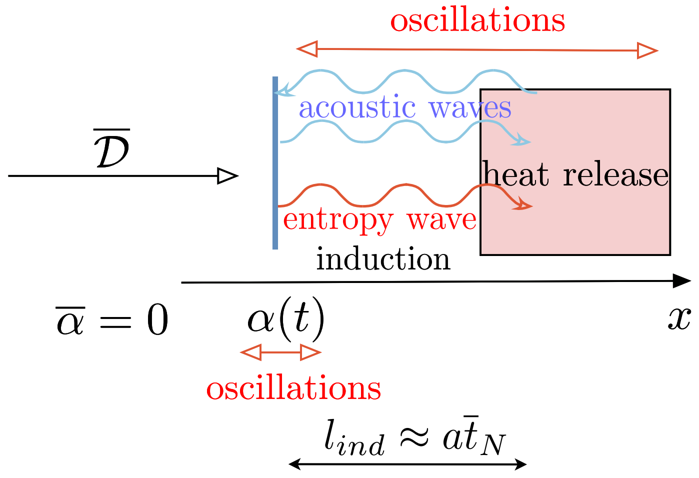
anticipating $\quad \breve{\theta} = O(\epsilon^2) \quad a/a_u = 1 + O(\epsilon^2) \quad 1 - \breve{u} = O(\epsilon) \quad \breve{\pi} = O(\epsilon)$

the variation of $a$ is negligible in $\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]p \pm \frac{1}{a}\left[\frac{\partial}{\partial t} + (u \pm a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{\overline{t}_N}$

$$\left[\frac{\partial}{\partial \mathrm{t}} + (1+\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2\dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma-1)\breve{\pi}] = \epsilon^2\dot{\mathrm{w}}$$

$$\left[\frac{\partial}{\partial \mathrm{t}} - (1-\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} - \breve{u}) = \epsilon^2\dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}},$$

# Slow time scale




$M_u^2 - 1 = O(\epsilon) \Rightarrow$ transonic flow: $u/a = 1 + O(\epsilon)$

*time scale of the downstream propagating acoustic wave :* $l_{ind}/a = \bar{t}_N$

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]p + \frac{1}{a}\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

*time scale of the upstream propagating acoustic wave :* $l_{ind}/(a-u) \approx \bar{t}_N/\epsilon$

$$\frac{1}{\gamma p}\left[\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right]p - \frac{1}{a}\left[\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right]u = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

longest delay in the feed back loop $\approx \bar{t}_N/\epsilon$

## *Scaling*

$$\boxed{\text{Period of oscillation} = O(\bar{t}_N/\epsilon)}$$

non dimensional time of order unity $\quad \boxed{\tau \equiv \frac{t}{\bar{t}_N/\epsilon}} = \epsilon\,\mathrm{t} \quad \leftarrow \quad \mathrm{t} \equiv t/\bar{t}_N$

instantaneous position of the lead shock wave $\quad x = \alpha(\epsilon\, t/\bar{t}_N) \qquad \mathrm{a} \equiv \alpha(\tau)/(a_u\bar{t}_N) \leftarrow$ non dimensional position

non dimensional variable of order unity $\boxed{\mu,\ \pi,\ \theta :}$ $\quad \breve{u} \equiv \frac{u}{a_u} = 1 + \epsilon\mu, \quad \breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{\hat{p}}{\hat{p}_u}\right) = \epsilon\pi, \quad \breve{\theta} \equiv \frac{(T - \hat{T}_u)}{T_u} = \epsilon^2\theta$

reference frame of the moving shock : $\quad \tau \equiv \epsilon\mathrm{t}, \qquad \xi \equiv \mathrm{x} - \mathrm{a}(\tau), \qquad \frac{\partial}{\partial \mathrm{x}} = \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial \mathrm{t}} = \epsilon\left(\frac{\partial}{\partial \tau} - \dot{\mathrm{a}}_\tau\frac{\partial}{\partial \xi}\right)$

$\mathrm{x} \equiv x/(a_u\bar{t}_N)$

⚠ $u$ and $\mu$ are flow velocities in the lab frame

$$\left[\frac{\partial}{\partial \mathrm{t}} + (1+\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2\dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma-1)\breve{\pi}] = \epsilon^2\dot{\mathrm{w}}$$

$$\left[\frac{\partial}{\partial \mathrm{t}} - (1-\breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} - \breve{u}) = \epsilon^2\dot{\mathrm{w}}, \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}},$$

$\Rightarrow$

$$\boxed{\begin{array}{ll} \dfrac{\partial(\pi+\mu)}{\partial \xi} = 0 & \dfrac{\partial(\theta - h\pi - \psi)}{\partial \xi} = 0, \\ \left[\dfrac{\partial}{\partial \tau} + (\mu - \dot{\mathrm{a}}_\tau)\dfrac{\partial}{\partial \xi}\right](\pi - \mu) = \dot{\mathrm{w}}, & \dfrac{\partial \psi}{\partial \xi} = \dot{\mathrm{w}} \end{array}}$$

$h \equiv (\gamma - 1)/\epsilon = O(1)$

$\dot{\mathrm{a}}_\tau \equiv \mathrm{da}(\tau)/\mathrm{d}\tau$

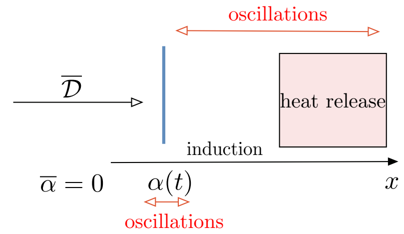
Arrhenius law : $\quad \dot{\mathrm{w}}(\psi,\theta) = (1-\psi)\mathrm{e}^{\beta_e(\theta - \overline{\theta}_N)}$ with $\boxed{\beta_e \equiv \frac{E}{k_B T_N}\epsilon^2 = O(1)}$

# Asymptotic model for CJ or near CJ regimes

$\epsilon^2 \equiv q_m/c_pT_u \ll 1, \quad (\gamma-1)=O(\epsilon), \quad E/k_BT_N = O(1/\epsilon^2)$ (Clavin Williams 2002)

$$\frac{\partial(\pi+\mu)}{\partial\xi}=0 \qquad \frac{\partial(\theta-h\pi-\psi)}{\partial\xi}=0,$$

$$\left[\frac{\partial}{\partial\tau}+(\mu-\dot{a}_\tau)\frac{\partial}{\partial\xi}\right](\pi-\mu)=\dot{w}, \qquad \frac{\partial\psi}{\partial\xi}=\dot{w}$$




$$\dot{a}_\tau \equiv da(\tau)/d\tau$$

$$M_u=(\overline{\mathcal{D}}-\dot{\alpha}_t)/a_u=\overline{M}_u-\epsilon\,\dot{a}_\tau$$

## *Boundary conditions at the Neumann state*

$h\equiv(\gamma-1)/\epsilon=O(1) \qquad f\equiv$ overdrive factor

$$\frac{T_N}{T_u}\approx 1+(\gamma-1)(M_u^2-1),$$

$$\frac{p_N}{p_u}\approx\frac{\rho_N}{\rho_u}\approx 1+(M_u^2-1)$$

$$M_u^2-1\approx 2(\overline{M}_u-1)-2\epsilon\,\dot{a}_\tau$$

$$(\overline{M}_u-1)\approx\varepsilon\sqrt{f}$$

$$\rho_u(\overline{\mathcal{D}}-\dot{\alpha}_t)=\rho_N(u|_{x=\alpha}-\dot{\alpha}_t)$$

$u$ and $\mu$ are flow velocities in the lab frame

$$\xi=0: \quad \theta\equiv\theta_N=2h(\sqrt{f}-\dot{a}_\tau), \quad \pi\equiv\pi_N=2(\sqrt{f}-\dot{a}_\tau)$$

$$\theta_N-h\pi_N=0$$

$$\xi=0: \quad \mu\equiv\mu_N=-\sqrt{f}+2\dot{a}_\tau \quad \text{and} \quad \psi=0,$$

$$\mu_N+\pi_N=\sqrt{f}$$

$$\mu+\pi=\sqrt{f} \qquad \theta=h\sqrt{f}-h\mu+\psi$$

The problem is reduced to solve two equations for $\mu$ and $\psi$

$$\boxed{\begin{array}{c}\left[\frac{\partial}{\partial\tau}+(\mu-\dot{a}_\tau)\frac{\partial}{\partial\xi}\right]\mu=-\frac{\dot{w}}{2}, \qquad \frac{\partial\psi}{\partial\xi}=\dot{w}(\psi,\theta)\\ \xi=0: \quad \mu=-\sqrt{f}+2\,\dot{a}_\tau \quad \text{and} \quad \psi=0,\end{array}}$$

## *Boundary condition in the burnt gas*

$$\boxed{\xi\to\infty: \qquad \psi=1, \qquad \mu=\overline{\mu}_b=-\sqrt{f-1}}$$

yields an integral equation for $\dot{a}_\tau(\tau)$

$$\dot{\mathrm{w}}(\psi = 1) = 0$$

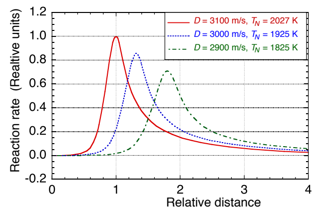
Nonlinear equation for a transonic reacting flow

$$\left[\frac{\partial}{\partial \tau} + (\mu - \dot{\mathrm{a}}_\tau)\frac{\partial}{\partial \xi}\right]\mu = -\frac{\dot{\mathrm{w}}}{2}, \qquad \frac{\partial \psi}{\partial \xi} = \dot{\mathrm{w}}(\psi, \theta)$$

$$\theta = h\sqrt{f} - h\mu + \psi$$

$$\xi = 0: \quad \mu = -\sqrt{f} + 2\,\dot{\mathrm{a}}_\tau \quad \text{and} \quad \psi = 0, \qquad \xi \to \infty: \quad \psi = 1, \quad \mu = \overline{\mu}_b = -\sqrt{f - 1}$$

## Result for simplified chemical kinetics




*Simplification:*

The reaction rate depends only on $T_N(t)$

The stability analysis is similar to that of strongly overdriven regimes !

Similar integral equation but with a delay controlled by the upstream running acoustic wave

$$\Delta(\xi) = \int_0^\xi \frac{\mathrm{d}\xi}{|\mu(\xi)|}$$

$$\boxed{\dot{\mathrm{a}}_\tau(\tau) = \int_0^\infty \left[\frac{1}{4\sqrt{f}}\Omega'_N(\xi) + G(\xi)\right]\dot{\mathrm{a}}_\tau(\tau - \Delta(\xi))\mathrm{d}\xi}$$

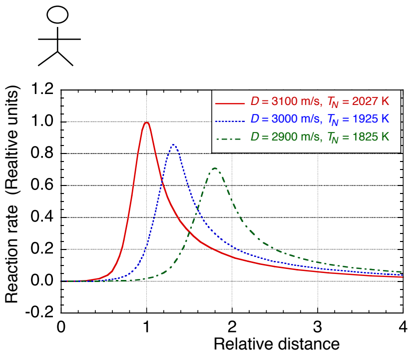
Instability due to thermal sensitivity

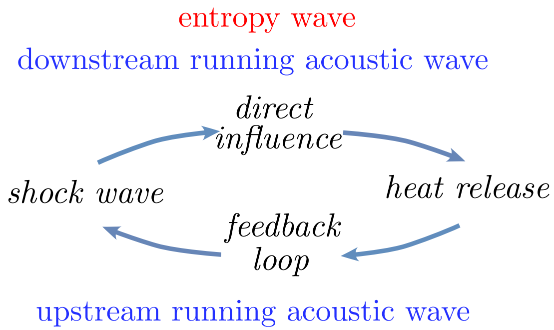
Stabilizing term due to residual compressible effects

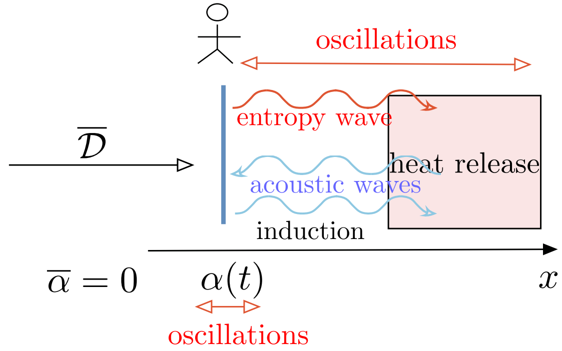
Reduced propagation velocity of the shock front

# GENERAL CONCLUSION

Galloping detonations are due to a phase shift in the loop between the lead shock and the heat release, controlled by the entropy wave and the upstream running acoustic wave




*Strongly overdriven detonation in the Newtonian limit* ( P.C. & L. He 1996)

quasi-isobaric flow

dominant mechanism: entropy wave →

*CJ (or near CJ) conditions close to the instability threshold* ( P.C. & F.A. Williams 2002)

transonic flow

dominant mechanism: acoustic wave ←

Comparison with DNS

Tofaili, Lodato, Vervisch and Clavin *Combustion and Flame* (2021) Vol. **232** 111535

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases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ECM & CNRS


## Lecture XIII
## Stability analysis of shock waves

# Lecture 13 : Stability analysis of shock waves

## 13-1. Acoustic waves and entropy-vorticity wave

*Linearized Euler equations*

*Linearized flow field*

## 13-2. Analyses

*Dispersion relation for general materials*

*Classification of normal modes*

*Spontaneous emission of sound and instability*

*Stability of shocks in ideal gases*

*Stability of reacting shocks*

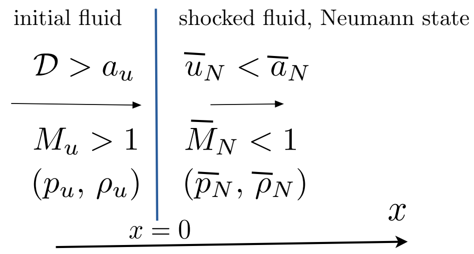
# Acoustic waves and entropy-vorticity wave

Shock wave $\approx$ hydrodynamic discontinuity + Rankine-Hugoniot conditions

thickness $\approx$ mean free path

in the frame of the unperturbed planar shock

$M = \mathcal{D}/a$




The flow of shocked gas in a planar wave is uniform

$\mathcal{D} > a_u \Rightarrow$ the upstream flow in a wrinkled shock is not perturbed

supersonic wave

Flow velocity $\overline{u}_N$ is sufficiently large $\Rightarrow$ the diffusive fluxes are negligible:

compressed gas:

$$Ds/Dt = 0 \quad \Leftrightarrow \quad \partial s/\partial t + \underline{u}.\underline{\nabla} s = 0$$

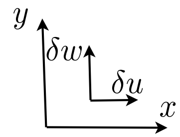
no entropy production in the compressed gas

The entropy of shocked gas is modified at the Neumann state of a wrinkled shock

$$\Rightarrow \nabla s(\mathbf{r}, t) \neq 0$$

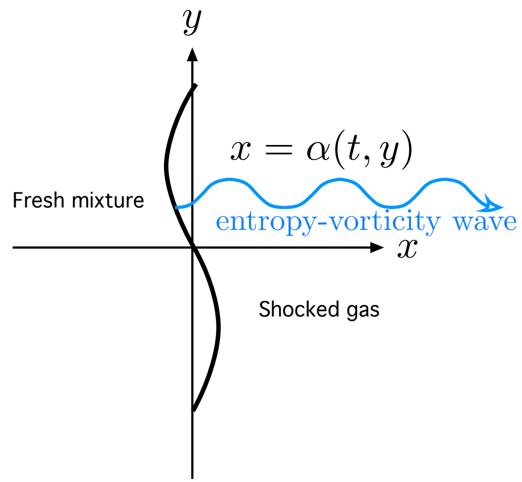
entropy production inside the shock thickness: entropy jump !

# Linearized Euler equations




(written in 2-D for simplicity. Extension to 2-D is straightforward)

$$u = \overline{u}_N + \delta u, \quad w = \delta w, \quad \rho = \overline{\rho}_N + \delta\rho, \quad p = \overline{p}_N + \delta p$$




*Compressed gas*

$$\frac{1}{\overline{\rho}_N}\frac{\mathrm{D}}{\mathrm{D}t}\delta\rho + \frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta w = 0,$$

$\mathrm{D}/\mathrm{D}t = \partial/\partial t + \overline{u}_N \partial/\partial x$

$$\overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta u = -\frac{\partial}{\partial x}\delta p, \qquad \overline{\rho}_N \frac{\mathrm{D}}{\mathrm{D}t}\delta w = -\frac{\partial}{\partial y}\delta p,$$

$\partial p/\partial\rho|_{s=\mathrm{cst}} \equiv a$

$$\frac{\mathrm{D}}{\mathrm{D}t}\delta s = 0 \Rightarrow \frac{\mathrm{D}}{\mathrm{D}t}\delta p = \overline{a}_N^2 \frac{\mathrm{D}}{\mathrm{D}t}\delta\rho,$$

isentropic: no entropy production but propagation of entropy from the shock

*Wave equation for the pressure (d'Alembert equation)*

eliminating $\delta\rho \quad \Rightarrow \quad \frac{1}{\overline{\rho}_N \overline{a}_N^2}\frac{\mathrm{D}}{\mathrm{D}t}\delta p + \frac{\partial}{\partial x}\delta u + \frac{\partial}{\partial y}\delta w = 0$

eliminating $\delta u$ and $\delta w \quad \Rightarrow \quad \boxed{\frac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p - \overline{a}_N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta p = 0}$

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity $\overline{u}_N$

# Linearized flow field

## *Flow splitting*

acoustic wave + vorticity wave

$$\delta p = \delta p^{(a)}, \quad \boxed{\begin{array}{l}\delta u = \delta u^{(a)} + \delta u^{(i)} \\ \delta w = \delta w^{(a)} + \delta w^{(i)}\end{array}}$$

$$\boxed{\bar{\rho}_N \left[\frac{\partial}{\partial t} + \underline{u}_N.\underline{\nabla}\right] \underline{u}^{(a)} = -\underline{\nabla} p}$$

the pressure fluctuations are fully propagated by acoustic waves in the shocked gas moving at constant velocity $\overline{u}_N$

## *Wave equation for the pressure (d'Alembert equation)*

eliminating $\delta\rho \quad \Rightarrow \quad \dfrac{1}{\bar{\rho}_N \bar{a}_N^2} \dfrac{\mathrm{D}}{\mathrm{D}t} \delta p + \dfrac{\partial}{\partial x} \delta u^{(a)} + \dfrac{\partial}{\partial y} \delta w^{(a)} = 0$

eliminating $\delta u$ and $\delta w \quad \Rightarrow \quad \boxed{\dfrac{\mathrm{D}^2}{\mathrm{D}t^2} \delta p - \bar{a}_N^2 \left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) \delta p = 0}$

## *Entropy-vorticity wave (isobaric)*

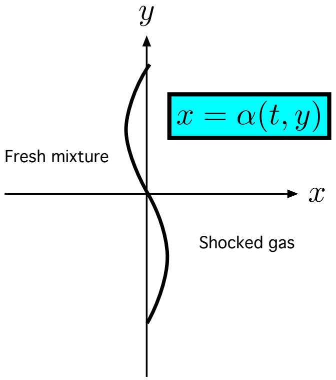
$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \overline{u}_N \frac{\partial}{\partial x}$$

$$\begin{array}{l}\mathrm{D}\delta u^{(i)}/\mathrm{D}t = 0 \\ \mathrm{D}\delta w^{(i)}/\mathrm{D}t = 0\end{array} \quad \Leftrightarrow \quad \begin{array}{l}\dfrac{\partial \delta u^{(i)}}{\partial t} + \overline{u}_N \dfrac{\partial \delta u^{(i)}}{\partial x} = 0 \\ \dfrac{\partial \delta w^{(i)}}{\partial t} + \overline{u}_N \dfrac{\partial \delta w^{(i)}}{\partial x} = 0\end{array} \quad \Leftrightarrow \quad \boxed{\begin{array}{l}\delta u^{(i)}(y, t - x/\overline{u}_N) \\ \delta w^{(i)}(y, t - x/\overline{u}_N)\end{array}}$$

$$\frac{1}{\bar{\rho}_N \bar{a}_N^2} \cancel{\frac{\mathrm{D}}{\mathrm{D}t}} \delta p + \frac{\partial}{\partial x} \delta u^{(i)} + \frac{\partial}{\partial y} \delta w^{(i)} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x} \delta u^{(i)} + \frac{\partial}{\partial y} \delta w^{(i)} = 0 \Rightarrow \boxed{\frac{\partial}{\partial t} \delta u^{(i)} = \overline{u}_N \frac{\partial}{\partial y} \delta w^{(i)}}$$

# Linearized flow field

## *Normal-mode analysis*




$$\alpha(y,t)=\tilde{\alpha}\mathrm{e}^{iky+\sigma t}$$

$$\boxed{\delta p(x,y,t)=\tilde{p}(x)\mathrm{e}^{iky+\sigma t}}$$

$k\in\mathrm{Re}$ $\quad\sigma\in\mathrm{Z}$ $\quad\boxed{\sigma(k)\ ?}$

$$\delta p=\tilde{p}_N\exp\left(\mathrm{i}l_\pm x+\mathrm{i}ky+\sigma t\right)\qquad \tilde{p}(x)=\tilde{p}_N\mathrm{e}^{il_\pm x}$$

2nd-order algebraic eq.

$$l_\pm(\sigma,k)\qquad(\sigma+\mathrm{i}l_\pm\overline{u}_N)^2+\overline{a}_N^2(l_\pm^2+k^2)=0\quad\Longleftarrow\quad\frac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p-\overline{a}_N^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta p=0\quad\text{wave eq.}$$

No length-scale other than $|k|^{-1}$ in the problem

$$\mathrm{i}\frac{l_\pm}{|k|}=\frac{\overline{M}_NS\pm\sqrt{1+S^2}}{\sqrt{1-\overline{M}_N^2}}\quad\text{with}\quad\boxed{S\equiv\frac{\sigma}{\overline{a}_N|k|}\frac{1}{\sqrt{1-\overline{M}_N^2}},}$$

$$\overline{\rho}_N\frac{\mathrm{D}}{\mathrm{D}t}\delta u=-\frac{\partial}{\partial x}\delta p,\qquad \overline{\rho}_N\frac{\mathrm{D}}{\mathrm{D}t}\delta w=-\frac{\partial}{\partial y}\delta p,\quad\Rightarrow$$

$$\tilde{u}^{(a)}=-\frac{\mathrm{i}l_\pm\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\mathrm{e}^{\mathrm{i}l_\pm x},$$

$$\tilde{w}^{(a)}=-\frac{\mathrm{i}k\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\mathrm{e}^{\mathrm{i}l_\pm x},$$

$$x=0:\ \delta p=\delta p_N(y,t)=\tilde{p}_N\mathrm{e}^{\mathrm{i}ky+\sigma t}$$

↖ RH conditions

$$\tilde{u}^{(i)}=\left[\tilde{u}_N+\frac{\mathrm{i}l_\pm\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\right]\mathrm{e}^{-\sigma x/\overline{u}_N},$$

$$\tilde{w}^{(i)}=\left[\tilde{w}_N+\frac{\mathrm{i}k\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\right]\mathrm{e}^{-\sigma x/\overline{u}_N},$$

$$\Leftrightarrow\quad\boxed{\begin{array}{c}\delta u^{(i)}(y,t-x/\overline{u}_N)\\ \delta w^{(i)}(y,t-x/\overline{u}_N)\end{array}}$$

Rankine-Hugoniot conditions

## *Incompressibility condition*

$$\frac{\partial}{\partial x}\delta u^{(i)}=-\frac{\partial}{\partial y}\delta w^{(i)}\Rightarrow\frac{\partial}{\partial t}\delta u^{(i)}=\overline{u}_N\frac{\partial}{\partial y}\delta w^{(i)}\Longrightarrow$$

$$\boxed{-\frac{(\mathrm{i}l_\pm\sigma+\overline{u}_Nk^2)}{(\sigma+\mathrm{i}l_\pm\overline{u}_N)}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}-\sigma\frac{\tilde{u}_N}{\overline{u}_N}+\mathrm{i}k\tilde{w}_N=0}$$

# Lecture 13 : Stability analysis of shock waves

## 13-1. Acoustic waves and entropy-vorticity wave

*Linearized Euler equations*

*Linearized flow field*

## 13-2. Analyses

*Dispersion relation for general materials*

*Classification of normal modes*

*Spontaneous emission of sound and instability*

*Stability of shocks in ideal gases*

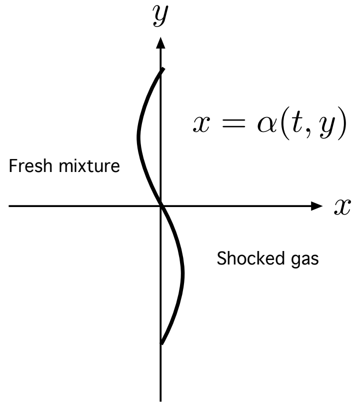
*Stability of reacting shocks*

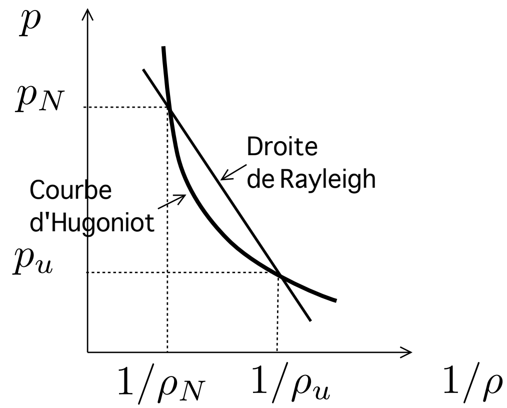
# Dispersion relation for general materials

*Rankine Hugoniot relations (general material)*

$M = \mathcal{D}/a$

| initial fluid | shocked fluid, Neumann state |
|---|---|
| $\mathcal{D} > a_u$ $\longrightarrow$ | $u_N < a_N$ $\longrightarrow$ |
| $M_u > 1$ | $M_N < 1$ |
| $(p_u,\ \rho_u)$ | $(p_N,\ \rho_N)$ |




$$\alpha(t, y) = \hat{\alpha}\, \mathrm{e}^{\sigma t + iky}$$

non-dimensional parameters

$$r \equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0,$$

$$\overline{M}_N \leftrightarrow n \equiv \frac{\overline{\rho}_N}{\rho_u} \frac{\overline{M}_N^2}{\left(1 - \overline{M}_N^2\right)}$$

$$\delta p_N = \frac{1}{r} \left(\frac{\rho_u}{\overline{\rho}_N}\right)^2 \overline{\mathcal{D}}^2 \delta \rho_N$$

2 parameters for the material: $r$ and $n$

*Linear rate* Quadratic equation for $\sigma^2/\overline{a}_N^2 k^2$ $\Leftarrow$ $\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n$

$$aS^4 + 2bS^2 + c = 0, \qquad S^2 \equiv \frac{\sigma^2}{\overline{a}_N^2 k^2} \frac{1}{(1 - M_N^2)}$$

$$a \equiv (1+r)^2 - 4\overline{M}_N^2, \quad b \equiv (1-r^2)n - 2\overline{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

P. Clavin and F.A. Williams, *Philos. Trans. R. Soc. London*, **A, 370,** 597-624

# Classification of normal modes

$$\alpha(t,y) = \hat{\alpha}\,\mathrm{e}^{\sigma t + iky} \qquad \delta p = \tilde{p}_N \exp\left(\mathrm{i}l_{\pm}x + \mathrm{i}ky + \sigma t\right)$$

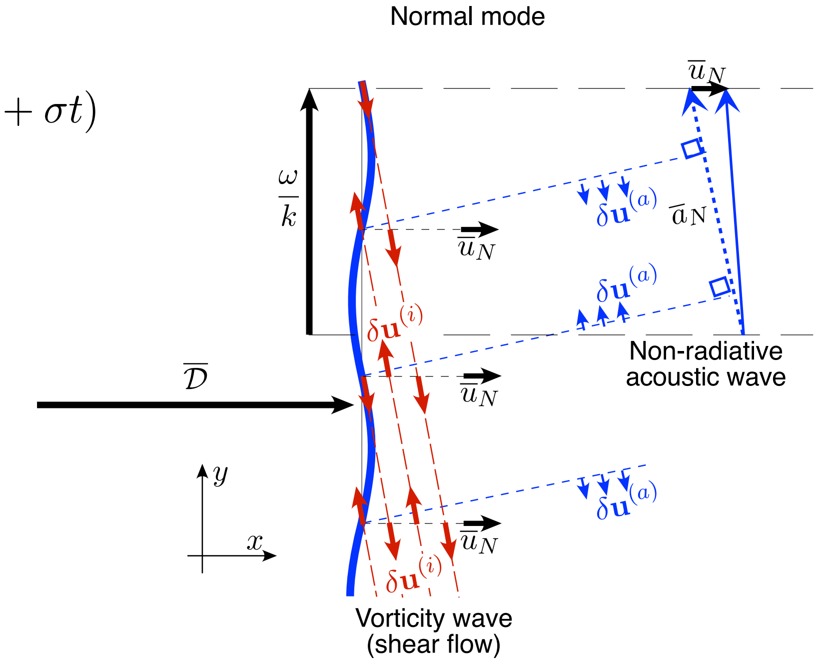
$$\boxed{aS^4 + 2bS^2 + c = 0,} \qquad \boxed{S^2 \equiv \frac{\sigma^2}{\overline{a}_N^2 k^2}\frac{1}{(1-M_N^2)}}$$

$$a \equiv (1+r)^2 - 4\overline{M}_N^2, \quad b \equiv (1-r^2)n - 2\overline{M}_N^2, \quad c \equiv (1-r)^2 n^2 > 0.$$

$\mathrm{Re}(\sigma) < 0$ : stable mode exponentially damped $\boxed{\mathrm{e}^{-|\mathrm{Re}(\sigma)|t}}$

$\mathrm{Re}(\sigma) > 0$ : unstable mode exponentially amplified $\boxed{\mathrm{e}^{|\mathrm{Re}(\sigma)|t}}$

$S^2 < 0$ : $\quad \mathrm{Re}(\sigma) = 0, \quad \omega \equiv \mathrm{Im}(\sigma) \neq 0 \quad$ neutral oscillatory modes




longitudinal component of the velocity (unperturbed shock) of the sound wave:

$$\delta\mathbf{u} = \delta\mathbf{u}^{(i)} + \delta\mathbf{u}^{(a)} \qquad \boxed{\mathbf{e}_x.(\overline{u}_N\mathbf{e}_x - \overline{a}_N\mathbf{e}_K) = \overline{u}_N - \overline{a}_N\frac{l}{\sqrt{l^2+k^2}}}$$

$\mathrm{Re}(\sigma) = 0$ Neutral oscillatory modes

Spontaneous generation of sound. Radiating condition: $\overline{u}_N\sqrt{l^2+k^2} - \overline{a}_N l > 0$

Non-radiating condition: $\overline{u}_N\sqrt{l^2+k^2} - \overline{a}_N l < 0$

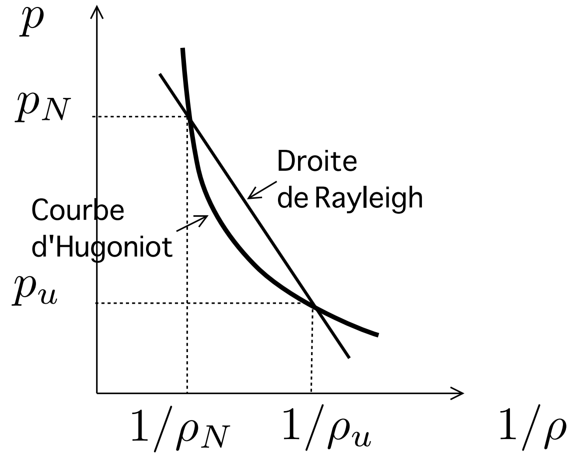
unstable $\boxed{t^n}$ $n > 0$

stable $\boxed{1/t^n}$

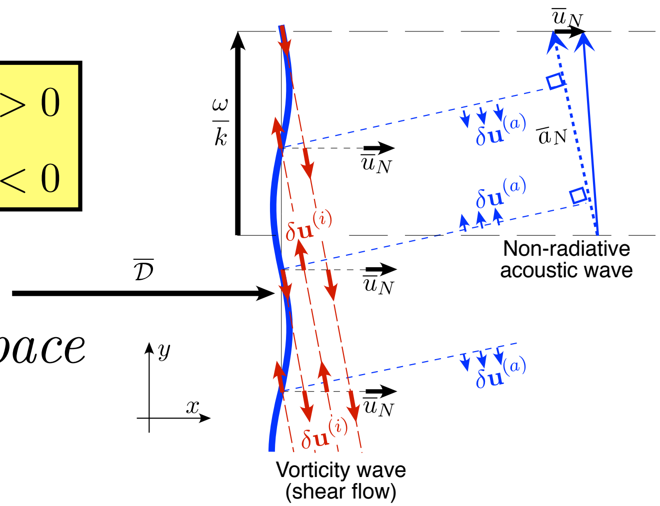
# Classification of normal modes

2 non-dimensional parameters

$$\boxed{r \equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0, \qquad n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)}}$$




Normal mode

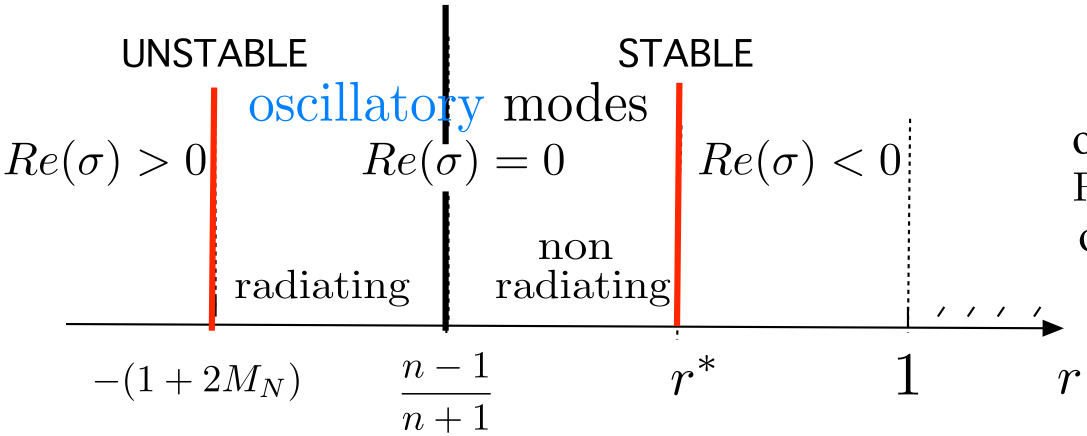



Neutral oscillatory modes

Spontaneous generation of sound. Radiating condition: $\overline{u}_N\sqrt{l^2+k^2}-\overline{a}_N l > 0$

Non-radiating condition: $\overline{u}_N\sqrt{l^2+k^2}-\overline{a}_N l < 0$

*Classification of the normal modes in the parameters space*




oscillatory unstable $\boxed{t^n}$

$\mathrm{Re}(\sigma) = 0$ $\quad n > 0$

oscillatory stable $\boxed{1/t^n}$

$$r^* = \frac{n - \sqrt{\left(1-M_N^2\right)\left(1-\frac{\rho_u}{\rho_N}\right)}}{n+1}$$

P. Clavin and F.A. Williams, *Philos. Trans. R. Soc. London*, **A, 370,** 597-624

# Stability of shocks in ideal gases

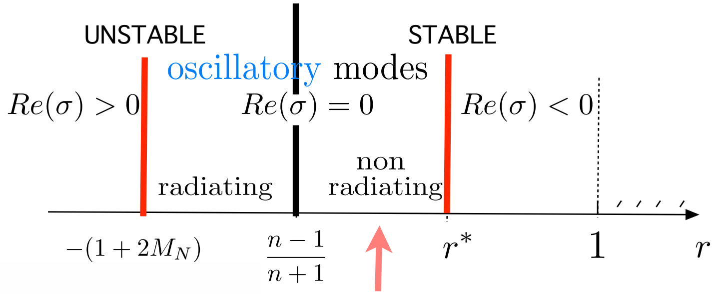
polytropic gas, $\gamma = \text{cst.}$ $\quad \dfrac{u_N}{\mathcal{D}} = \dfrac{\rho_u}{\rho_N} = \dfrac{(\gamma-1)M_u^2+2}{(\gamma+1)M_u^2}, \quad \dfrac{p_N}{p_u} = \dfrac{2\gamma M_u^2-(\gamma-1)}{(\gamma+1)}, \quad M_N^2 = \dfrac{(\gamma-1)M_u^2+2}{2\gamma M_u^2-(\gamma-1)}$

$$r \equiv -\frac{(\rho_u \overline{\mathcal{D}})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} = \frac{1}{\overline{M}_u^2}, \qquad n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)} = \frac{\overline{M}_u^2}{(\overline{M}_u^2-1)}$$

$$\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2+(1-r)n, \implies \pm 2S\overline{M}_N\sqrt{1+S^2} = S^2\left(1+\overline{M}_u^{-2}\right)+1$$

$$\overline{M}_u > 1, \quad \gamma > 1 \quad \Rightarrow \quad \boxed{(n-1)/(n+1) < r < r*}$$

$$r^* = \frac{n-\sqrt{(1-M_N^2)\left(1-\frac{\rho_u}{\rho_N}\right)}}{n+1} \qquad \frac{1}{2\overline{M}_u^2-1} < \frac{1}{\overline{M}_u^2} < \frac{\overline{M}_u^2-(\overline{M}_u^2-1)^2\sqrt{2\overline{M}_u^{-2}[2\gamma\overline{M}_u^2-(\gamma-1)]^{-1}}}{2\overline{M}_u^2-1}$$




Clavin Williams 2012

Shock waves in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws $1/t^{3/2}$

OK with experiments K.C. Lapworth (1959) *J.F.M*, **6,** 469-480

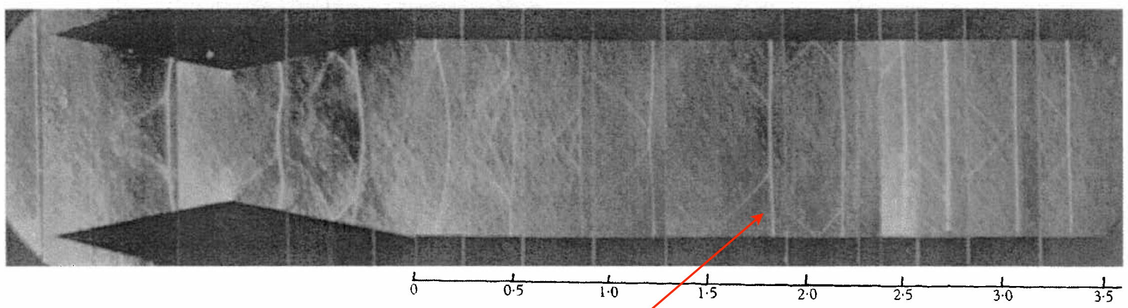



FIGURE 6 (plate 1). Photographs of shock wave at $M_s = 1{\cdot}41$.

Formation of Mach stems
(see next lecture)

# Stability of reacting shocks

P. Clavin and F.A. Williams, (2012) *Philos. Trans. R. Soc. London*, **A, 370,** 597-624

Reacting shocks = detonations considered as an hydrodynamic discontinuity

thickness = 0: no modification of the inner structure

$$r \equiv -\frac{\rho_u \mathcal{D}^2}{\mathrm{d}p_b/\mathrm{d}\rho_b^{-1}} = \overline{M}_u^2 \frac{1+\overline{\mathcal{V}}_b}{1-\overline{M}_u^2\overline{\mathcal{V}}_b} \qquad \overline{\mathcal{V}}_b = \left[\overline{M}_u^{-2} - (1+\chi)\right]/2 \qquad \chi \equiv \sqrt{\left(1-\overline{M}_u^{-2}\right)^2 - 4\mathcal{Q}\overline{M}_u^{-2}} \qquad r = \frac{(1-\chi)+\frac{1}{\overline{M}_u^2}}{(1+\chi)+\frac{1}{\overline{M}_u^2}}$$

$$0 \leqslant \chi < 1$$

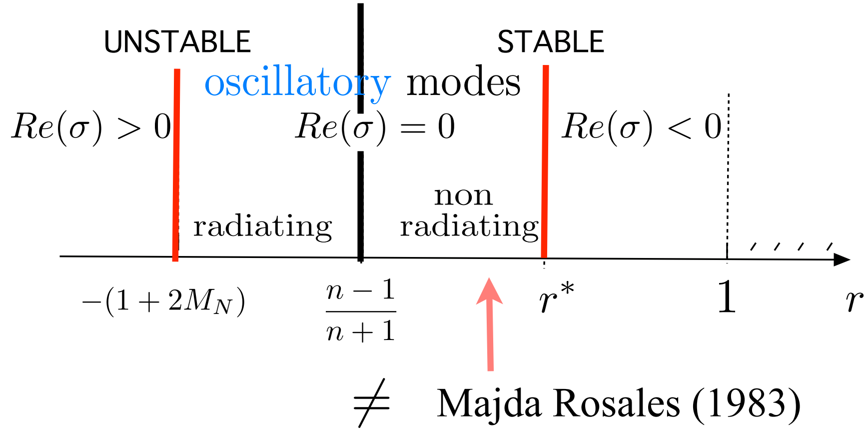
CJ wave ↗ overdriven regime

$$\chi = \sqrt{\left(1-\overline{M}_u^{-2}\right)^2 - \left(1-\overline{M}_{u_{CJ}}^{-2}\right)^2 \overline{M}_{u_{CJ}}^2 \overline{M}_u^{-2}}$$

$$n \equiv \left(\frac{\overline{\rho}_b}{\rho_u}\right)\frac{\overline{M}_b^2}{1-\overline{M}_b^2} == \frac{1}{\overline{M}_u^{-2} - 1 - 2\overline{\mathcal{V}}_b} = \frac{1}{\chi}$$

$$r = \frac{(1-\chi)+\frac{1}{\overline{M}_u^2}}{(1+\chi)+\frac{1}{\overline{M}_u^2}} = \left(\frac{n-1}{n+1}\right)\left[\frac{1+\frac{1}{\overline{M}_u^2(1-\chi)}}{1+\frac{1}{\overline{M}_u^2(1+\chi)}}\right]$$

$$\boxed{\left(\frac{n-1}{n+1}\right) \leqslant r}$$




Overdriven reacting shocks in polytropic gases have neutral modes with non-radiating acoustic waves

They are stable with a relaxation of initial disturbances in power laws

For the CJ marginal regime the acoustic waves in the burned gas propagate in the direction parallel to the unperturbed planar solution

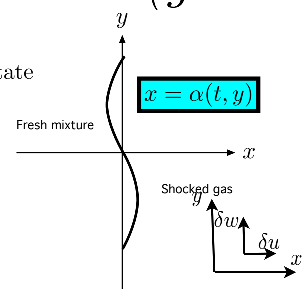
# Details of the calculation

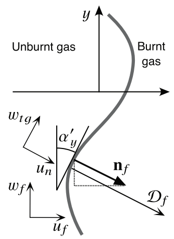
(details of the calculation)

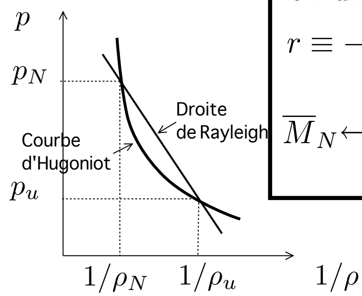
# *Rankine Hugoniot relations (general material)*

$M = \mathcal{D}/a$

| initial fluid | shocked fluid, Neumann state |
|---|---|
| $\mathcal{D} > a_u$ | $u_N < a_N$ |
| $M_u > 1$ | $M_N < 1$ |
| $(p_u,\ \rho_u)$ | $(p_N,\ \rho_N)$ |




non-dimensional parameters

$$r \equiv -\frac{(\rho_u \mathcal{D})^2}{\mathrm{d}\overline{p}_N/\mathrm{d}\overline{\rho}_N^{-1}} > 0,$$

$$\overline{M}_N \leftrightarrow n \equiv \frac{\overline{\rho}_N}{\rho_u}\frac{\overline{M}_N^2}{\left(1-\overline{M}_N^2\right)}$$

$$\boxed{\delta p_N = \frac{1}{r}\left(\frac{\rho_u}{\overline{\rho}_N}\right)^2 \overline{\mathcal{D}}^2 \delta\rho_N}$$

## Jump conditions

p.5 lecture IV

mass $\quad \rho_N(u_N - \partial\alpha/\partial t - w_N \partial\alpha/\partial y) = \rho_u(\overline{\mathcal{D}} - \partial\alpha/\partial t) \quad \Rightarrow \quad \delta\rho_N \overline{u}_N + \overline{\rho}_N(\delta u_N - \partial\alpha/\partial t) = -\rho_u \partial\alpha/\partial t$

$$\boxed{\delta m = -\rho_u \frac{\partial\alpha}{\partial t}}$$

tangential momentum $\quad w_N = (\overline{\mathcal{D}} - u_N)\alpha'_y \quad \Rightarrow \quad \delta w_N = (\overline{\mathcal{D}} - \overline{u}_N)\partial\alpha/\partial y$

tangent to the Hugoniot curve (geometrical construction) $\quad \Rightarrow \quad \dfrac{\delta p_N}{\overline{p}_N} = \dfrac{1}{r}\dfrac{(\rho_u \mathcal{D})^2}{\overline{p}_N\overline{\rho}_N}\dfrac{\delta\rho_N}{\overline{\rho}_N}$

longitudinal momentum $\quad p_N - p_u = -m^2\left(\dfrac{1}{\rho_N} - \dfrac{1}{\rho_u}\right)$

$$\left.\begin{array}{l} \dfrac{\delta p_N}{\overline{p}_N} = \dfrac{2}{m}\left(1 - \dfrac{p_u}{\overline{p}_N}\right)\rho_u \dfrac{\partial\alpha}{\partial t} + \dfrac{\overline{m}^2}{\overline{p}_N\overline{\rho}_N}\dfrac{\delta\rho_N}{\overline{\rho}_N} \end{array}\right\} \Rightarrow$$

$$\frac{\delta p_N}{\overline{p}_N} = -2\frac{\left(1 - \frac{p_u}{\overline{p}_N}\right)}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}, \qquad \frac{\delta\rho_N}{\overline{\rho}_N} = -2\left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{r}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}$$

$$\frac{\delta u_N}{\overline{u}_N} = \left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{1+r}{(1-r)}\frac{\partial\alpha/\partial t}{\overline{\mathcal{D}}}, \qquad \frac{\delta w_N}{\overline{u}_N} = \left(\frac{\overline{\rho}_N}{\rho_u} - 1\right)\frac{\partial\alpha}{\partial y}$$

# *Linear rate*

$$\pm\sqrt{S^2+1}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{a}_N\overline{u}_N} = -S\frac{\tilde{u}_N}{\overline{u}_N} + \frac{\mathrm{i}k\tilde{w}_N}{|k|\overline{u}_N}\frac{\overline{M}_N}{\sqrt{1-\overline{M}_N^2}}$$

$$\alpha(y,t) = \hat{\alpha}\mathrm{e}^{\mathrm{i}ky+\sigma t} \qquad \frac{\partial\alpha}{\partial t} = \sigma\alpha \qquad \frac{\partial\alpha}{\partial y} = \mathrm{i}k\alpha$$

$$\Rightarrow \quad \boxed{\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n \qquad S \equiv \frac{\sigma}{\overline{a}_N|k|}\frac{1}{\sqrt{1-\overline{M}_N^2}}}$$

taking the square $\quad \boxed{aS^4 + 2bS^2 + c = 0,} \qquad S^2 \equiv \dfrac{\sigma^2}{\overline{a}_N^2 k^2}\dfrac{1}{(1-M_N^2)}$

$$a \equiv (1+r)^2 - 4\overline{M}_N^2, \qquad b \equiv (1-r^2)n - 2\overline{M}_N^2, \qquad c \equiv (1-r)^2 n^2 > 0.$$

Quadratic equation for $\sigma^2/\overline{a}_N^2 k^2$



Clavin Williams 2012

(details of the calculation)

# Dispersion relation for general materials

*Compatibility condition*

$$-\frac{(\mathrm{i}l_\pm\sigma+\overline{u}_N k^2)}{\cancel{(\sigma+\mathrm{i}l_\pm\overline{u}_N)}}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}-\sigma\frac{\tilde{u}_N}{\overline{u}_N}+\mathrm{i}k\tilde{w}_N=0$$

$$\left(\frac{\partial}{\partial t}+\overline{u}_N\frac{\partial}{\partial x}\right)^2\delta p-\overline{a}_N^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\delta p=0 \qquad (\sigma+\mathrm{i}l_\pm\overline{u}_N)^2+\overline{a}_N^2(l_\pm^2+k^2)=0 \;\Rightarrow\; \sigma^2+2\mathrm{i}\sigma l_\pm\overline{u}_N-l_\pm^2\overline{u}^2+\overline{a}_N^2(l_\pm^2+k^2)=0$$

$$-(\mathrm{i}l_\pm\overline{u}_N\sigma+\overline{u}_N^2k^2)=\sigma^2+\mathrm{i}l_\pm\sigma\overline{u}_N+(\overline{a}_N^2-\overline{u}_N^2)(l_\pm^2+k^2)$$

$$\boxed{\mathrm{i}\frac{l_\pm}{|k|}=\frac{\overline{M}_N S\pm\sqrt{1+S^2}}{\sqrt{1-\overline{M}_N^2}}} \qquad \overline{a}_N^2(l_\pm^2+k^2)=-(\sigma+\mathrm{i}l_\pm\overline{u}_N)^2 \quad \text{with} \quad S\equiv\frac{\sigma}{\overline{a}_N|k|}\frac{1}{\sqrt{1-\overline{M}_N^2}}$$

$$\Rightarrow \quad =(\sigma+\mathrm{i}l_\pm\overline{u}_N)\left[\sigma-\left(1-\overline{M}_N^2\right)(\sigma+\mathrm{i}l_\pm\overline{u}_N)\right]$$

$$\Rightarrow \quad =(\sigma+\mathrm{i}l_\pm\overline{u}_N)\left[\sigma\overline{M}_N^2-\left(1-\overline{M}_N^2\right)\mathrm{i}l_\pm\overline{u}_N\right]$$

$$-(\mathrm{i}l_\pm\overline{u}_N\sigma+\overline{u}_N^2k^2)=-\cancel{(\sigma+\mathrm{i}l_\pm\overline{u}_N)}\sqrt{1-\overline{M}_N^2}\left[\pm\sqrt{1+S^2}\right]|k|\overline{u}_N$$

$$-\frac{(\mathrm{i}l_\pm\sigma+\overline{u}_N k^2)}{(\sigma+\mathrm{i}l_\pm\overline{u}_N)}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}-\sigma\frac{\tilde{u}_N}{\overline{u}_N}+\mathrm{i}k\tilde{w}_N=0 \;\Rightarrow\; -\sqrt{1-\overline{M}_N^2}\left[\pm\sqrt{1+S^2}\right]|k|\overline{u}_N\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}-\sigma\frac{\tilde{u}_N}{\overline{u}_N}+\mathrm{i}k\tilde{w}_N=0$$

$$\pm\sqrt{S^2+1}\boxed{\frac{\tilde{p}_N}{\overline{\rho}_N\overline{a}_N\overline{u}_N}}+S\boxed{\frac{\tilde{u}_N}{\overline{u}_N}}-\frac{\mathrm{i}k}{|k|}\boxed{\frac{\tilde{w}_N}{\overline{u}_N}}\frac{\overline{M}_N}{\sqrt{1-\overline{M}_N^2}}=0$$

*(Buckmaster Ludford 1988. Clavin et al. 1997)*

The Rankine Hugoniot relations yields an equation for $S\propto\dfrac{\sigma}{\overline{a}_N|k|}$

$$\frac{\delta p_N}{\overline{p}_N}\propto\frac{\dot{\alpha}_t}{\overline{u}_N},\quad \boxed{\frac{\tilde{p}_N}{\overline{p}_N}\propto\mathrm{i}\sigma\frac{\hat{\alpha}}{\overline{u}_N}} \qquad \frac{\delta u_N}{\overline{u}_N}\propto\frac{\dot{\alpha}_t}{\overline{u}_N},\quad \boxed{\frac{\tilde{u}_N}{\overline{u}_N}\propto\mathrm{i}\sigma\frac{\hat{\alpha}}{\overline{u}_N}} \qquad \frac{\delta w_N}{\overline{u}_N}\propto\alpha'_y,\quad \boxed{\frac{\tilde{w}_N}{\overline{u}_N}\propto\mathrm{i}k\hat{\alpha}}$$

*Downstream boundary condition*

$x\to\infty$ : bounded condition (in the unstable case, $\mathrm{Re}\sigma>0$)

$$\delta p=\tilde{p}_N\exp\left(\mathrm{i}l_\pm x+\mathrm{i}ky+\sigma t\right)$$

selection of the the sign in $l_\pm$ such that $e^{il_\pm x}$ does not diverge

# Spontaneous emission of sound and instability

D'Yakov Kontorovich 1954-57

## *Oscillatory neutral modes*

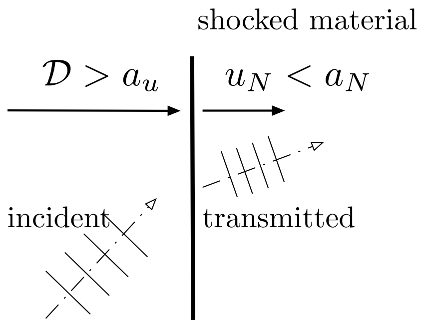
$\pm 2\overline{M}_N S\sqrt{1+S^2} = (1+r)S^2 + (1-r)n$

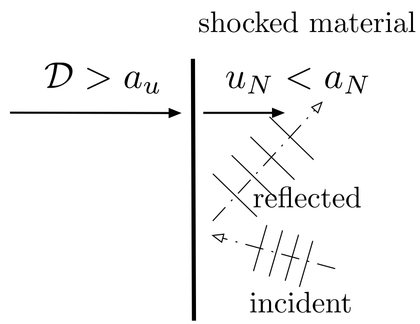
neutral oscillatory mode

$S = i\Omega, \quad \Omega > 1$

$$\Rightarrow \begin{cases} \text{radiating waves:} \quad l/|k| = \left[\overline{M}_N\Omega - \sqrt{\Omega^2-1}\right]/\sqrt{1-\overline{M}_N^2}, /\sqrt{1-\overline{M}_N^2}, \\ 2\overline{M}_N\Omega\sqrt{\Omega^2-1} = -\Omega^2(1+r) + (1-r)n > 0, \\ \text{non-radiating waves:} \quad l/|k| = \left[\overline{M}_N\Omega + \sqrt{\Omega^2-1}\right]/\sqrt{1-\overline{M}_N^2}, \\ -2\overline{M}_N\Omega\sqrt{\Omega^2-1} = -\Omega^2(1+r) + (1-r)n < 0 \end{cases}$$

## *Transmitted or reflected sound wave*




If a normal mode is radiating the response of the shock diverges when the reflected (or transmitted) waves matched the radiating normal mode

$$\begin{matrix}\text{reflected} & \longrightarrow & \tilde{p}_r \\ \text{incident} & \longrightarrow & \tilde{p}_i\end{matrix} = -\frac{\left[2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n\right]}{\left[-2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n\right]} \quad \longleftarrow \quad \text{denominator goes trough 0}$$

A neutral oscillatory mode that is radiating is considered as unstable

D'Yakov Kontorovich 1954-57

## *Power laws of neutral modes*

Square root in the dispersion relation $\pm 2\overline{M}_N\Omega\sqrt{\Omega^2-1} - \Omega^2(1+r) + (1-r)n = 0$ $\Longrightarrow$ cut in the complex plane $\Longrightarrow$ power laws

Bates 2007

Damping ($n < 0$) or amplification ($n > 0$) involving power laws $t^n$
can be described by Laplace transform not by Fourier transform

Neutral modes with non-radiating acoustic waves relaxe follc ing a power law in time $\quad t^n \quad n < 0$

Neutral modes with a radiating acoustic wave is unstable according a power law in time $\quad t^n \quad n > 0$

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

**Paul Clavin**

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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

## Lecture XIV
## Nonlinear dynamics of shock waves.
## Triple point and Mach stem formation.

# Lecture 14: Nonlinear dynamics of shock waves Mach stem formation

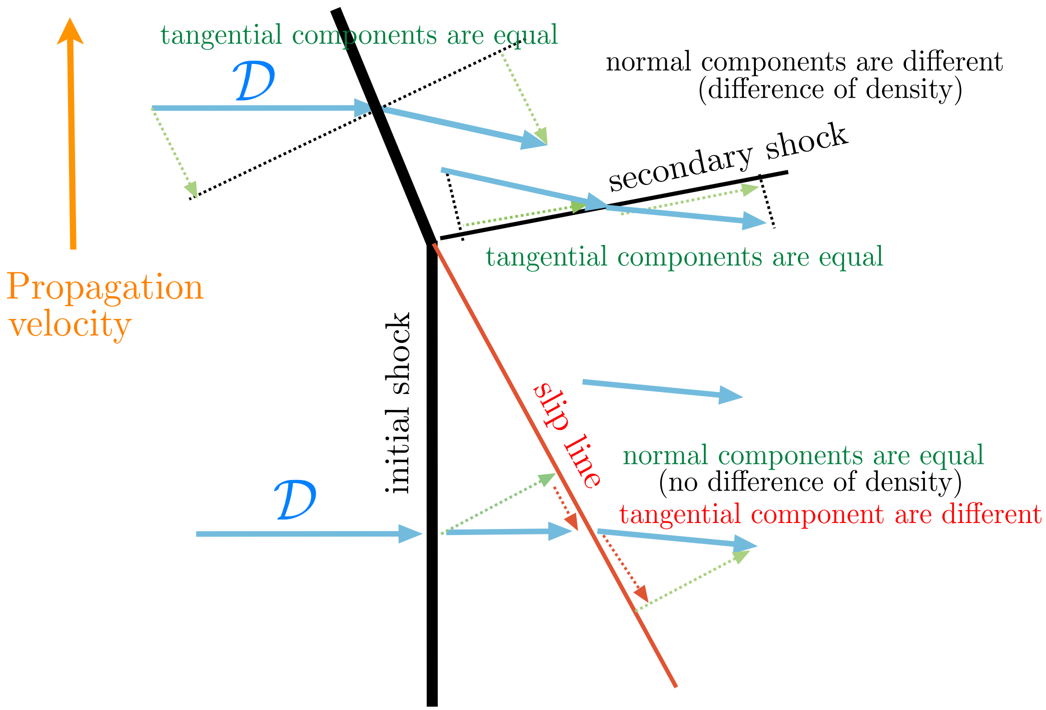
# Introduction.
# Recent experimental and DNS results

## What is a Mach stem ?

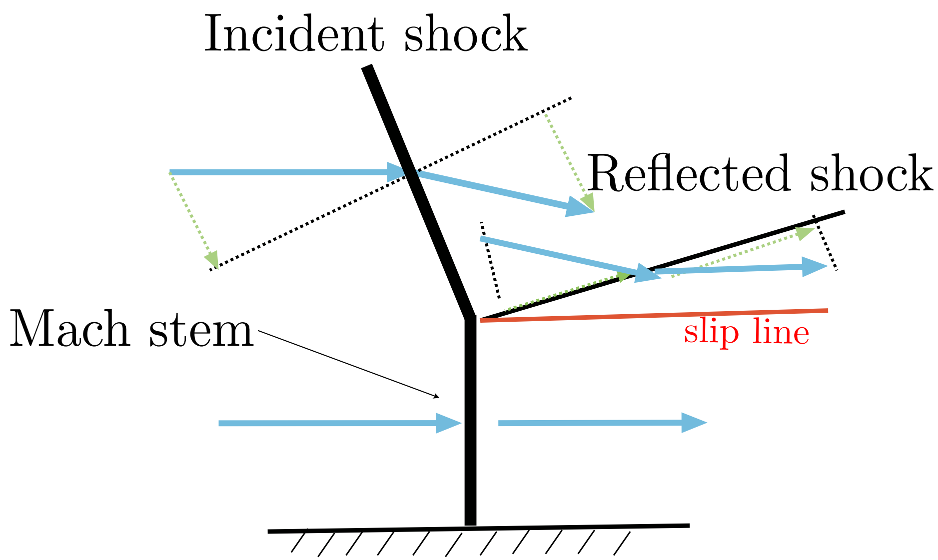
Triple point = 3 shock waves + 1 slip line (degenerescence of shear layer)

called also contact line discontinuity ( Courant Friedrichs 1948)

First observed during the reflection of an oblique shock front incident an a wall




example of a triple point propagating in a uniform flow



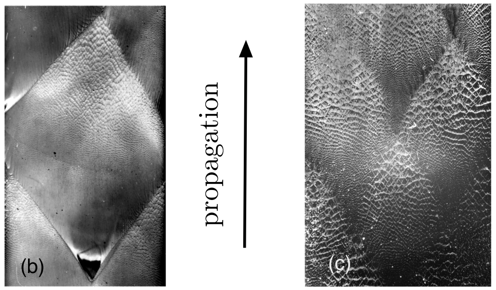

example of stationary triple point

# Mach stems and cellular detonations

## *Experimental observations of the cellular structure of detonations*

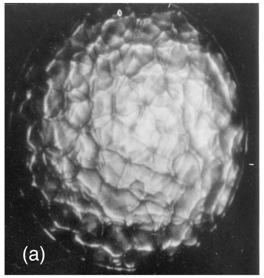
Transverse structures of gaseous detonations have been observed for a long time

Shchelkin and Troshin (1965) Mono Book Corp.




F. Joubert *et al.* (2008) *Combust. Flame*, **152**, 482-495

markings left on soot-coated foils on the walls




H.N. Presles *et al.* (1987) *Combust. Flame*, **70**, 207-213

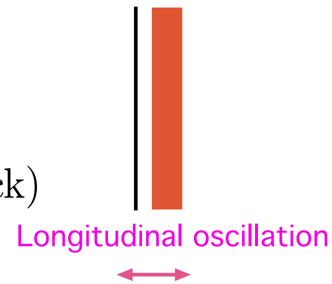
visualisation of the cellular structures by optical methods

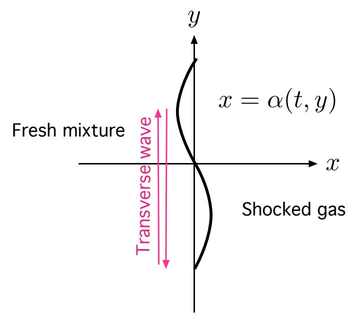
## *Underlying linear mechanisms*

longitudinal oscillation of the complex shock reaction zone
(Galloping detonation) *see lecture XII*

+

transverse oscillatory modes (normal modes of the lead shock)
*see lecture XIII*




## *Nonlinear mechanisms*

singularity of slope of the lead shock $\Rightarrow$ formation of Mach stems

*Theory:*

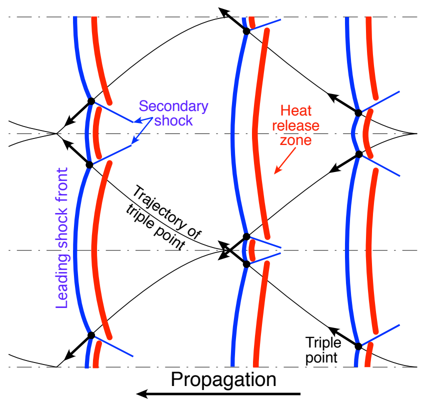
Clavin (2002) *Int. J. Bifurcation and Chaos.*, **12** (11) 2535-2546

Clavin and Denet (2002) *P.R.L* ***88*** (4) 044502

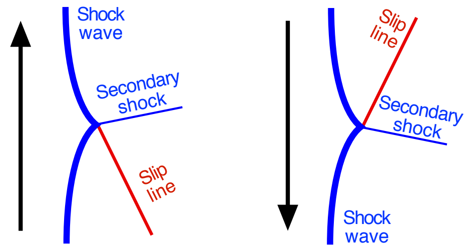
Clavin (2013) *J. Fluid Mech.*, **721**, 324-339

Clavin (2017) *Combust. Sci. Technol.*, **189** (5), 747-775

trajectory of triple points




Mach stems propagating in the transverse direction



# Spontaneous formation of Mach stems on shock fronts

*Schlieren experiments in shock tubes* K.C. Lapworth (1959) *J.F.M*, **6**, 469-480

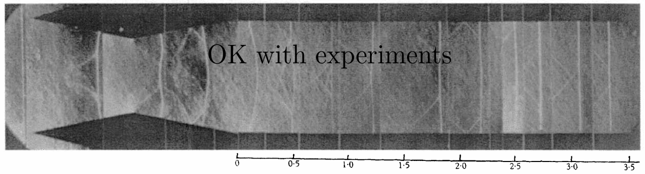


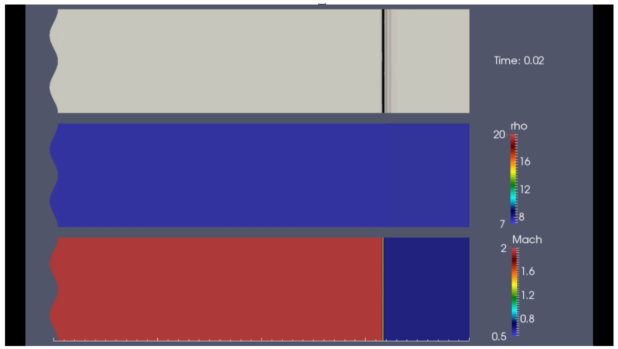

FIGURE 6 (plate 1). Photographs of shock wave at $M_s = 1{\cdot}41$.

*Shock reflection from a wavy wall*

M.G. Briscoe and A.A. Kovitz (1968) *J.F.M*, **31(3),** 529-546

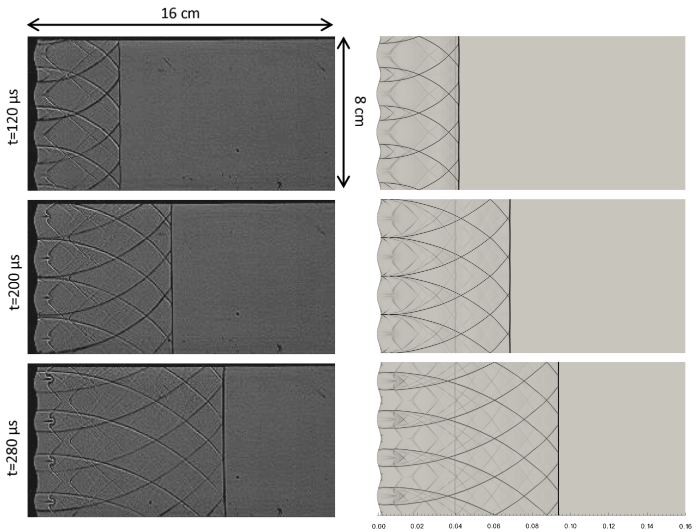
*DNS in 2 dimension geometry*

G. Lodato *et al.* (2016) *J.F.M*, **789**, 221-258

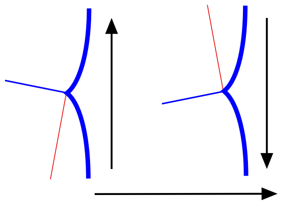



Comparison with experiments

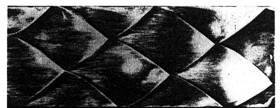
B. Denet *et al.* (2015) *Combust. Sci. Technol.* **187**, (1-2), 296-323




Triple points



quite similar to the markings left by the transverse structure of cellular detonations



Shchelkin Troshin 1965

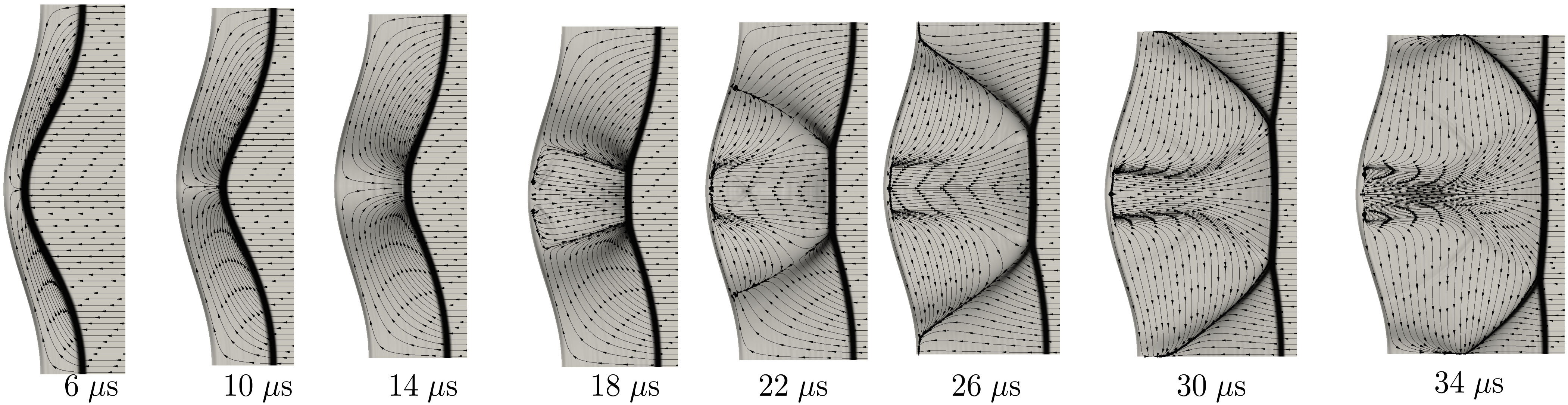
# *Spontaneous formation of Mach stems*

G. Lodato *et al.* (2016) *J.F.M*, **789**, 221-258

The incoming shock wave is not strong, $M_u = 1.5$ and the amplitude of wavy wall is small 1 mm

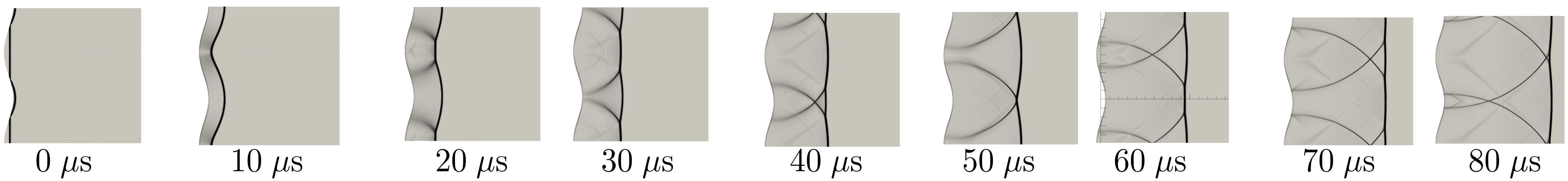
Immediately after reflection the wrinkled reflected shock has a smooth sinusoidal form

Singularity of slope is formed spontaneously at about 15 $\mu$s leading to triple points clearly observed as early as 20 $\mu$s




Lodato Vervisch 2015

Long-lived Mach stems on quasi-planar inert shock front




Lodato Vervisch 2015

Sufficiently far from the wall the wall effect becomes negligible

The shock is quasi-planar with Mach stems propagating in the transverse direction crossing each other without deformation as solitons are known to do

# Multidimensional analysis of shock fronts

*Analysis for strong shocks in the Newtonian limit*

P. Clavin (2017) *Combust. Sci. Technol.*, **189** (5), 747-775

## Linear dynamics

### *Distinguished limit*

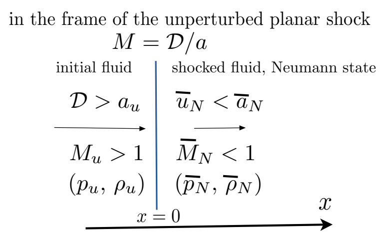
In order to simplify the presentation the analysis is performed for strong shock in the Newtonian limit

$$\overline{M}_u^2 \gg 1, \qquad (\gamma-1) \ll 1,$$

$$\overline{M}_u^2(\gamma-1) = O(1), \qquad \overline{M}_N^2 \approx (\gamma-1)/2 + 1/\overline{M}_u^2 \ll 1,$$

$$\boxed{\epsilon^2 \equiv M_N^2 \ll 1 \qquad \overline{M}_u = O(1/\epsilon), \qquad (\gamma-1) = O(\epsilon^2)}$$

$$\overline{u}_N/\overline{\mathcal{D}} \approx \epsilon^2, \qquad \overline{a}_N^2 \approx \overline{u}_N\overline{\mathcal{D}}, \qquad \overline{a}_N/a_u = O(1)$$




### *Rankine-Hugoniot relations* ( see p. 6 & p.9 lecture XIII)

$$\frac{\delta p_N}{\overline{p}_N} \approx -2\frac{\dot{\alpha}_t}{\overline{\mathcal{D}}}, \qquad \frac{\delta \rho_N}{\overline{\rho}_N} = -2\left(\frac{\overline{a}_u}{\overline{a}_N}\right)^2 \frac{\dot{\alpha}_t}{\overline{\mathcal{D}}},$$

$$(\delta u_N - \dot{\alpha}_t) = -\frac{[(\gamma-1)\overline{M}_u^2 - 2]}{2\overline{M}_u^2}\dot{\alpha}_t, \qquad \delta w_N \approx \overline{\mathcal{D}}\alpha'_y,$$

where for simplicity some unimportant $\epsilon^2$ terms have been omitted in $\delta p_N$ and $\delta w_N$

### *Quasi-isobaric approximation of the flow in the shocked gas*

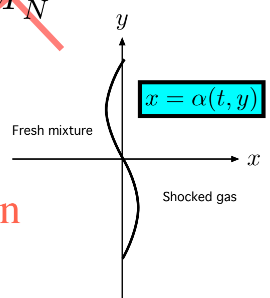
$$\pm 2S\cancel{\overline{M}_N}\sqrt{1+S^2} = S^2\left(1+\cancel{\overline{M}_u^{-2}}\right)+1 \qquad S \equiv \frac{\sigma}{\overline{a}_N|k|}\frac{1}{\sqrt{1-\cancel{\overline{M}_N^2}}} \quad \Rightarrow \quad \boxed{\sigma \approx \pm \mathrm{i}\overline{a}_N|k|}$$

Dispersion relation see p. 10 lecture XIII

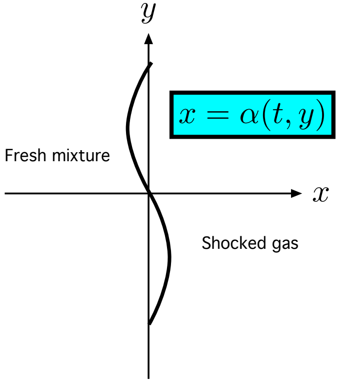
$$\omega \approx \overline{a}_N k, \qquad \boxed{\partial^2\alpha/\partial t^2 - \overline{a}_N^2\partial^2\alpha/\partial y^2 = 0}$$

$$x = \alpha(y,t)$$

Wave equation in the transverse direction



$$\boxed{\epsilon^2 \equiv M_N^2 \ll 1 \qquad \overline{M}_u = O(1/\epsilon), \qquad (\gamma-1) = O(\epsilon^2)}$$




$$\overline{u}_N/\overline{\mathcal{D}} \approx \epsilon^2, \qquad \overline{a}_N^2 \approx \overline{u}_N\overline{\mathcal{D}}, \qquad \overline{a}_N/a_u = O(1)$$

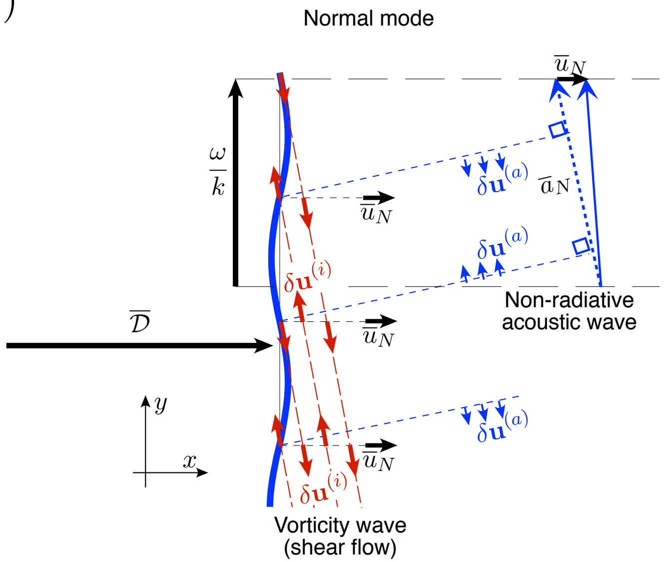
$$\omega \approx \overline{a}_N k, \qquad \boxed{\partial^2\alpha/\partial t^2 - \overline{a}_N^2 \partial^2\alpha/\partial y^2 = 0}$$

Wave equation

$$\omega \approx \overline{a}_N|k| \qquad \dot{\alpha}_t = O(\overline{a}_N\alpha'_y)$$

$$\boxed{\delta p = \tilde{p}_N \exp\left(\mathrm{i}l_\pm x + \mathrm{i}ky + \sigma t\right) \qquad il_\pm = \pm il,}$$

(see pp.4-5 lecture XIII)




$$(l/|k|)\sqrt{1-\overline{M}_N^2} = \overline{M}_N\Omega + \sqrt{\Omega^2-1} > 0 \quad \Rightarrow \quad \boxed{l/|k| = O(\epsilon)}$$

$$\overline{M}_N = \epsilon \qquad \Omega \approx \omega/(\overline{a}_N|k|) \approx 1$$

the acoustic waves propagates in a direction quasi-parallel to the front

$$\tilde{u}^{(a)} = -\frac{\mathrm{i}l_\pm\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\mathrm{e}^{\mathrm{i}l_\pm x}, \qquad \delta u^{(a)} = O(\epsilon\delta p_N/\overline{\rho}_N\overline{a}_N) \quad \delta u^{(a)} = O(\epsilon^2\delta p_N/\overline{\rho}_N\overline{u}_N)$$

see p. 4 lecture XIII

$$\tilde{w}^{(a)} = -\frac{\mathrm{i}k\overline{u}_N}{\sigma+\mathrm{i}l_\pm\overline{u}_N}\frac{\tilde{p}_N}{\overline{\rho}_N\overline{u}_N}\mathrm{e}^{\mathrm{i}l_\pm x}, \qquad \delta w^{(a)} = O(\delta p_N/\overline{\rho}_N\overline{a}_N) \quad \delta w^{(a)} = O(\epsilon\delta p_N/\overline{\rho}_N\overline{u}_N)$$

$$\Rightarrow \boxed{\begin{array}{l}\delta u^{(a)} = O(\epsilon^2\dot{\alpha}_t)\\ \delta w^{(a)} = O(\epsilon\dot{\alpha}_t)\end{array}}$$

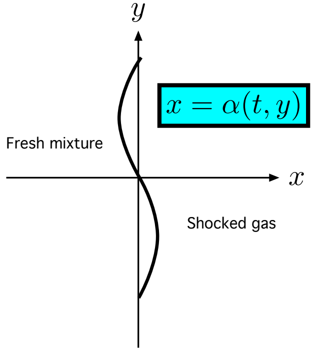
Rankine-Hugoniot:
( see p. 6 & p.10 lecture XIII)

$$\delta p_N/\overline{p}_N \approx -2\dot{\alpha}_t/\overline{\mathcal{D}} \Rightarrow \boxed{\delta p_N/(\overline{\rho}_N\overline{u}_N) = O(\dot{\alpha}_t)} \quad \nearrow$$

weak acoustic wave

$$\overline{p}_N \approx \overline{\rho}_N\overline{a}_N^2 \approx \overline{\rho}_N\overline{u}_N\frac{\overline{a}_N}{\overline{M}_N} \qquad \frac{\overline{a}_N}{\overline{M}_N} = \frac{\overline{a}_N/a_u}{\overline{M}_u\overline{M}_N}\overline{\mathcal{D}} = O(\overline{\mathcal{D}})$$

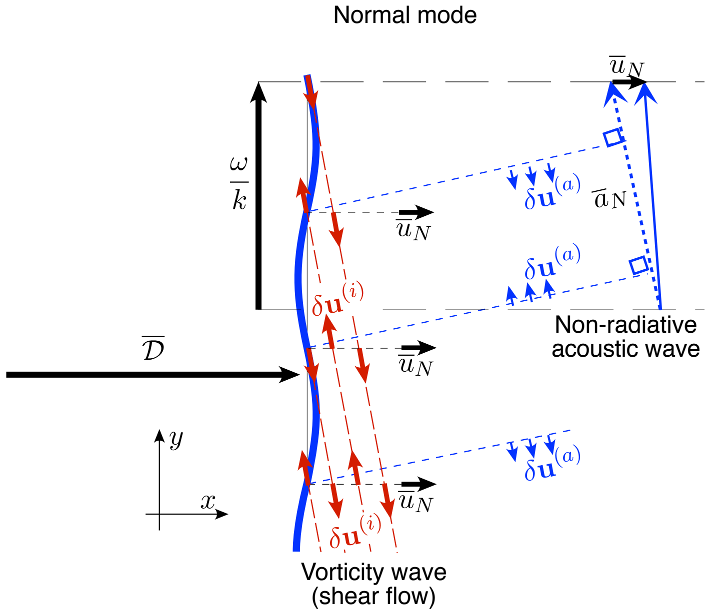
$$\boxed{\epsilon = M_N \ll 1 \qquad M_u = O(1/\epsilon), \qquad (\gamma - 1) = O(\epsilon)}$$




$$\overline{u}_N/\overline{\mathcal{D}} \approx \epsilon^2, \qquad \overline{a}_N^2 \approx \overline{u}_N \overline{\mathcal{D}}, \qquad \overline{a}_N/a_u = O(1)$$

$$\omega \approx \overline{a}_N k, \qquad \boxed{\partial^2\alpha/\partial t^2 - \overline{a}_N^2 \partial^2\alpha/\partial y^2 = 0}$$

Wave equation




Rankine-Hugoniot:

( see p. 6 & p.10 lecture XIII)

$$(\delta u_N - \dot{\alpha}_t) = -\frac{[(\gamma - 1)\overline{M}_u^2 - 2]}{2\overline{M}_u^2}\dot{\alpha}_t,$$

$$\Rightarrow \quad \boxed{\delta u_N \approx \dot{\alpha}_t \qquad \delta w_N \approx \overline{\mathcal{D}}\alpha'_y = O(\dot{\alpha}_t/\epsilon)}$$

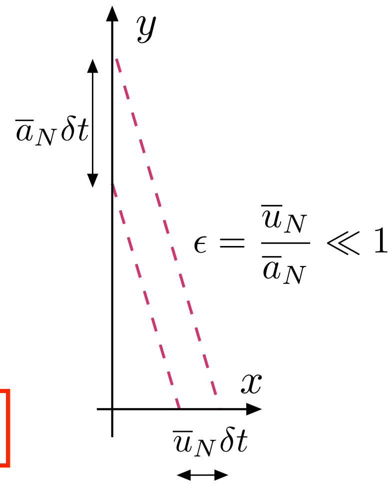
$$\delta u^{(i)}|_{x=0} = \delta u_N \approx \dot{\alpha}_t \qquad \delta w^{(i)}|_{x=0} = \delta w_N \approx \overline{\mathcal{D}}\alpha'_y$$

$$\Rightarrow \quad \boxed{\begin{array}{c} |\delta u^{(a)}/\delta u^{(i)}| = O(\epsilon^2) \\ |\delta w^{(a)}/\delta w^{(i)}| = O(\epsilon^2) \end{array}}$$

the acoustic waves are negligibly smaller than the vorticity wave

$$\boxed{|\delta u^{(i)}/\delta w^{(i)}| = O(\epsilon)}$$

the vorticity wave is a shear flow quasi-parallel to the front propagating at a subsonic velocity in the normal direction




$$\left(\frac{\partial}{\partial t} + \overline{u}_N\frac{\partial}{\partial x}\right)\mathbf{u}^{(i)} = 0 \quad \Rightarrow \quad \delta u^{(i)} = \dot{\alpha}_t(y, t - x/\overline{u}_N) \qquad \delta w^{(i)} = \overline{\mathcal{D}}\alpha'_y(y, t - x/\overline{u}_N)$$

$$\frac{\partial u^{(i)}}{\partial x} + \frac{\partial w^{(i)}}{\partial y} = 0 \quad \Rightarrow \quad -\frac{1}{\overline{u}_N}\frac{\partial^2\alpha}{\partial t^2} + \overline{\mathcal{D}}\frac{\partial^2\alpha}{\partial y^2} = 0$$

$$\boxed{\overline{u}_N\overline{\mathcal{D}} \approx \overline{a}_N^2 \quad \Rightarrow \quad \text{Wave equation}}$$

A subsonic wave that is sufficiently tilted yields a trace on the front that is sonic

# Weakly nonlinear analysis

P. Clavin (2013) *J. Fluid Mech.*, **721**, 324-339
P. Clavin (2017) *Combust. Sci. Technol.*, **189** (5), 747-775

## *Nonlinear Euler equations*

source terms

$$\frac{\partial u}{\partial t} + \overline{u}_N \frac{\partial u}{\partial x} = \mathcal{U} - \frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial w}{\partial t} + \overline{u}_N \frac{\partial w}{\partial x} = \mathcal{W} - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$-\mathcal{U} \equiv \delta u \frac{\partial u}{\partial x} + w\frac{\partial u}{\partial y}, \qquad -\mathcal{W} \equiv \delta u \frac{\partial w}{\partial x} + w\frac{\partial w}{\partial y},$$

where $\boxed{\begin{array}{l}\delta u \equiv u - \overline{u}_N \approx u^{(i)}\\ w \approx w^{(i)}\end{array}}$

$$\mathcal{U} = O(\varepsilon \partial u^{(i)}/\partial t) \qquad \mathcal{W} = O(\varepsilon \partial w^{(i)}/\partial t)$$

$\boxed{\begin{array}{l} u^{(i)} = \dot{\alpha}_t(y, t - x/\overline{u}_N)\\ w^{(i)} = \overline{\mathcal{D}}\alpha'_y(y, t - x/\overline{u}_N)\\ \overline{\mathcal{D}}\overline{u}_N \approx \overline{a}_N^2\\ \ddot{\alpha}_{tt} = \overline{a}_N^2 \alpha''_{y^2}\end{array}}$

Linear solution

where $\varepsilon \equiv |\dot{\alpha}_t|/\overline{u}_N = O(|\alpha'_y|\overline{a}_N/\overline{u}_N)$ that is $\boxed{\varepsilon = O(|\alpha'_y|/\epsilon)}$

The weakly nonlinear approximation is valid when the nonlinear terms $\mathcal{U}$ and $\mathcal{V}$ are small (compared with the linear terms)

This is the case for small amplitudes of the wrinkles $\boxed{|\alpha'_y| \ll \epsilon}$ where $\epsilon \equiv \overline{M}_N = \overline{u}_N/\overline{a}_N$ namely for $\varepsilon \ll 1$

## *Perturbation analysis for* $\varepsilon \ll 1$

$$\mathcal{U} \approx \frac{1}{2}\frac{\partial H}{\partial x}, \qquad \mathcal{W} \approx -\frac{1}{2}\frac{\overline{\mathcal{D}}}{\overline{u}_N}\frac{\partial H}{\partial y},$$

where $\quad H \equiv [-\dot{\alpha}_t^2(y, t - x/\overline{u}_N) + \overline{a}_N^2 {\alpha'_y}^2(y, t - x/\overline{u}_N)].$

progressive wave: $\dot{\alpha}_t = \pm \overline{a}_N \alpha'_y \quad \Rightarrow \quad H = 0, \quad \boxed{\mathcal{U} = 0, \quad \mathcal{W} = 0}$

no first order correction term coming from the Reynolds tensor

the shear wave $u^{(i)} = \dot{\alpha}_t(y, t - x/\overline{u}_N)$ $w^{(i)} = \overline{\mathcal{D}}\alpha'_y(y, t - x/\overline{u}_N)$ is an exact solution of the Euler equations for $p = 0$

the first order correction terms should come from the boundary conditions at $x = \alpha(y, t)$ (Rankine-Hugoniot)

Limiting the attention to the nonlinear corrections of order $\boxed{\varepsilon \equiv |\alpha'_y|/\epsilon}$ the Rankine-Hugoniot conditions yield

mass $\quad \rho_N(u_N - \partial\alpha/\partial t - w_N\partial\alpha/\partial y) = \rho_u(\overline{\mathcal{D}} - \partial\alpha/\partial t)$

p.5 lecture IV

tangential momentum $\quad \delta w_N = (\overline{\mathcal{D}} - \overline{u}_N)\partial\alpha/\partial y$

longitudinal momentum $\quad \dfrac{p_N}{p_u} = \dfrac{2\gamma M_u^2 - (\gamma-1)}{(\gamma+1)} \approx M_u^2$

where $\quad M_u = \dfrac{(\overline{\mathcal{D}} - \dot{\alpha}_t)}{a_u\left(1+\alpha'^2_y\right)^{1/2}}$

p.7 lecture X

$$\Rightarrow \boxed{\begin{array}{l} x=\alpha(y,t): \quad p=p_N, \quad u=u_N, \quad w=w_N \\ p_N/\overline{p}_N \approx 1-2\dot{\alpha}_t/\overline{\mathcal{D}}, \quad u_N-\overline{u}_N \approx \dot{\alpha}_t + \overline{\mathcal{D}}\alpha'^2_y, \quad w_N \approx \overline{\mathcal{D}}\alpha'_y \end{array}}$$

the only nonlinear term that yields a correction of order $\varepsilon \equiv |\alpha'_y|/\epsilon$

$$|\dot{\alpha}_t| = O(\overline{a}_N\alpha'_y), \quad \overline{\mathcal{D}}/\overline{a}_N = O(1/\epsilon) \qquad \Rightarrow \quad \overline{\mathcal{D}}|\alpha'^2_y/\dot{\alpha}_t| = O(|\alpha'_y|/\epsilon)$$

The shift of the front position also introduces quadratic terms

$$x=0: \quad \delta u \equiv u_f(y,t) \approx \delta u_N - \alpha u'_x, \quad w \equiv w_f(y,t) \approx w_N - \alpha w'_x$$

The nonlinear equations for the wrinkles is obtained from the incompressible condition

$$-\overline{u}_N^{-1}\partial u_f/\partial t + \partial w_f/\partial y = 0$$

$H=0 \quad \Rightarrow$ the nonlinear terms coming from shift of the front position do not contribute

$$-\overline{u}_N\partial(\alpha u'_x)/\partial t + \partial(\alpha w'_x)/\partial y = 0$$

$$-\frac{1}{\overline{u}_N}\frac{\partial u_N}{\partial t} + \frac{\partial w_N}{\partial y} = 0 \quad \Rightarrow \boxed{\frac{\partial^2\alpha}{\partial t^2} - \overline{a}_N^2\frac{\partial^2\alpha}{\partial y^2} + \overline{\mathcal{D}}\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial y}\right)^2 = 0} \qquad (\overline{a}_N^2 \approx \overline{\mathcal{D}}\overline{u}_N)$$

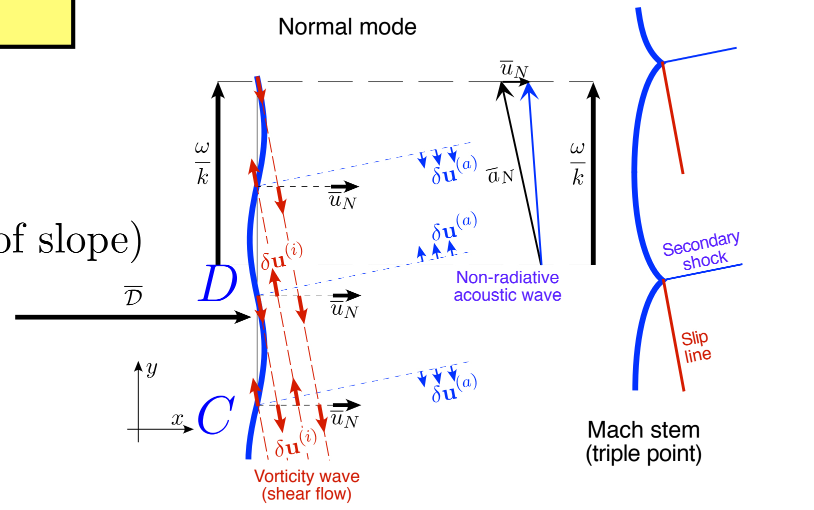
nonlinear correction of order $\varepsilon$, $\quad \overline{\mathcal{D}}\alpha'^2_y/|\dot{\alpha}_t| \approx (\overline{\mathcal{D}}/\overline{a}_N)|\alpha'_y| \approx \varepsilon$

# *Mach stem formation*

nonlinear correction of order $\varepsilon$, $\quad \overline{\mathcal{D}}\alpha_y'^2/|\dot{\alpha}_t| \approx (\overline{\mathcal{D}}/\bar{a}_N)|\alpha_y'| \approx \varepsilon$

$$\boxed{\frac{\partial^2\alpha}{\partial t^2} - \bar{a}_N^2\frac{\partial^2\alpha}{\partial y^2} + \overline{\mathcal{D}}\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial y}\right)^2 = 0}$$

P. Clavin (2013) *J. Fluid Mech.*, **721**, 324-339




Two timescales problem: wavelength of the wrinkles

short time $\tau_s \equiv L/\bar{a}_N$ (period of oscillation)

long time $\tau_l \equiv \tau_s/\varepsilon$ (for the formation of a singularity of slope)

Non-dimensional form $\quad \mathrm{t} \equiv t/\tau_s, \quad \mathrm{y} \equiv y/L, \quad \mathcal{A} \equiv \alpha/(\varepsilon\epsilon L)$

$$\frac{\partial^2\mathcal{A}}{\partial \mathrm{t}^2} - \frac{\partial^2\mathcal{A}}{\partial \mathrm{y}^2} + \varepsilon\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial\mathcal{A}}{\partial \mathrm{y}}\right)^2 = 0$$

small nonlinear correction term producing a singularity at finite (but long) non-dimensional time $1/\varepsilon$, $\mathrm{t} = O(1/\varepsilon)$

that is at the long timescale $t = O(\tau_l)$

so that $\mathcal{A}$ may be considered to depend on two reduced time variables $\boxed{\mathrm{t} \text{ and } \mathrm{t}' \equiv \varepsilon\mathrm{t},}$

$$\begin{cases} \dfrac{\partial}{\partial \mathrm{t}} \to \dfrac{\partial}{\partial \mathrm{t}} + \varepsilon\dfrac{\partial}{\partial \mathrm{t}'} \\[2mm] \dfrac{\partial^2}{\partial \mathrm{t}^2} \to \dfrac{\partial^2}{\partial \mathrm{t}^2} + 2\varepsilon\dfrac{\partial}{\partial \mathrm{t}}\dfrac{\partial}{\partial \mathrm{t}'} + \varepsilon^2\dfrac{\partial^2}{\partial \mathrm{t}'^2} \end{cases}$$

Considering a simple progressive wave $\boxed{\mathrm{y}' = \mathrm{y} \pm \mathrm{t}}$

and looking for a solution in the form $\mathcal{A}(\mathrm{y},\mathrm{t}) = A(\mathrm{y}',\mathrm{t}')$ one gets

$$2\varepsilon\frac{\partial^2 A}{\partial \mathrm{t}\partial \mathrm{t}'} + \cancel{\varepsilon^2\frac{\partial^2 A}{\partial \mathrm{t}'^2}} + \varepsilon\frac{\partial}{\partial \mathrm{t}}\left(\frac{\partial A}{\partial \mathrm{y}'}\right)^2 + \cancel{\varepsilon^2\frac{\partial^2}{\partial \mathrm{t}'}\left(\frac{\partial A}{\partial \mathrm{y}'}\right)^2} = 0$$

Burgers equation for $\quad B(y',t') \equiv \partial A/\partial y'$ $\quad \boxed{\partial B/\partial \mathrm{t}' + B\partial B/\partial \mathrm{y}' = 0}$

leading order $\quad 2\dfrac{\partial A}{\partial \mathrm{t}'} + \left(\dfrac{\partial A}{\partial \mathrm{y}'}\right)^2 \approx 0$

known to produce a singularity after a finite time see pp 3-4 lecture X

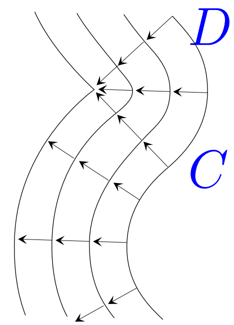
## *Geometrical construction for the slip line and the secondary shock*

Collapse of points C and D by a Huygens like construction

$\dot{\alpha}_t \approx \overline{\mathcal{D}}\alpha_y'^2 \quad \times \quad -\dot{\alpha}_t/(1+\alpha_y'^2)^{1/2} = \overline{\mathcal{D}}$

$\dot{\alpha}_t \approx \overline{\mathcal{D}}\alpha_y'^2/2$

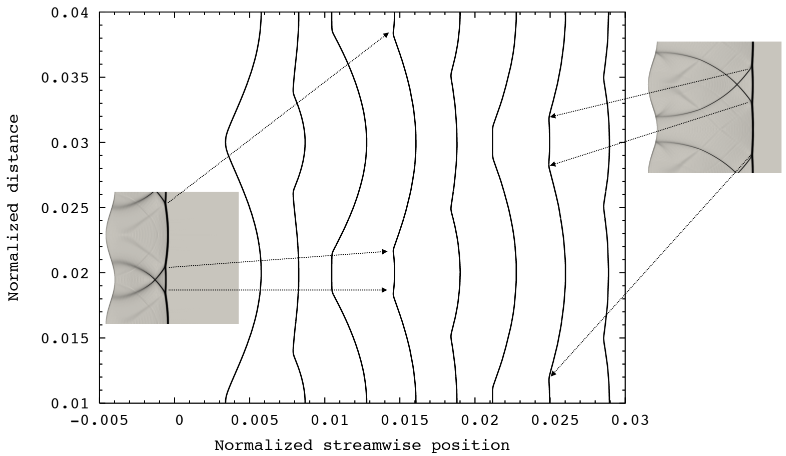
different only by a numerical factor 1/2



## *Numerical solution of the model equation and comparison with experiments*

$$\frac{\partial^2 \alpha}{\partial t^2} - \overline{a}_N^2 \frac{\partial^2 \alpha}{\partial y^2} + \overline{\mathcal{D}} \frac{\partial}{\partial t} \left( \frac{\partial \alpha}{\partial y} \right)^2 = 0$$

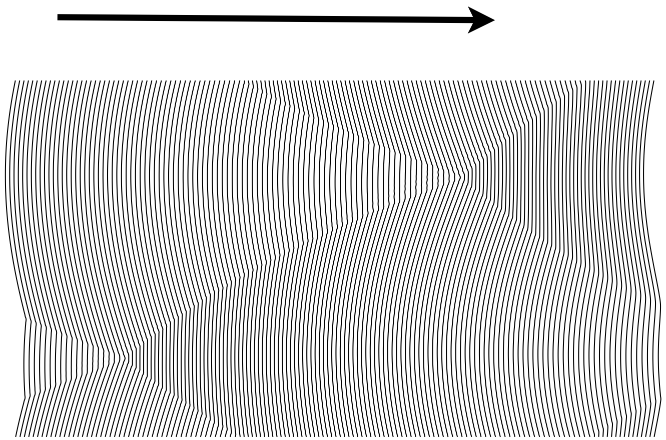
Initial condition: sinusoidal shock front of small amplitude




The formation of corners is clearly observed

Good agreement with experiments and DNS

G. Lodato *et al.* (2016) *J.F.M*, **789**, 221-258



The trajectory of the corners looks quite similar
to the traces left on the wall by a cellular detonation

B. Denet *et al.* (2015) *Combust. Sci. Technol.* **187**, (1-2), 296-323

# Shock-vortex interaction

Clavin (2013) *J. Fluid Mech.*, **721**, 324-339

## Formulation

strong shock + weak vortex

$$\epsilon^2 \equiv M_N^2 \ll 1 \qquad \overline{M}_u = O(1/\epsilon), \qquad (\gamma - 1) = O(\epsilon^2)$$

Consider a cylindrical and very subsonic vortex of diameter $L$ and turnover velocity $v_e$, $\boxed{v_e/\overline{a}_u \ll \epsilon}$ $(v_e \ll \overline{a}_u/\overline{M}_u)$

Interaction time $\tau_{int} = L/\overline{\mathcal{D}} \ll$ turnover time $L/v_e \Rightarrow$ frozen flow $u_e(\mathbf{r})\ w_e(\mathbf{r})$ + small disturbances of the front

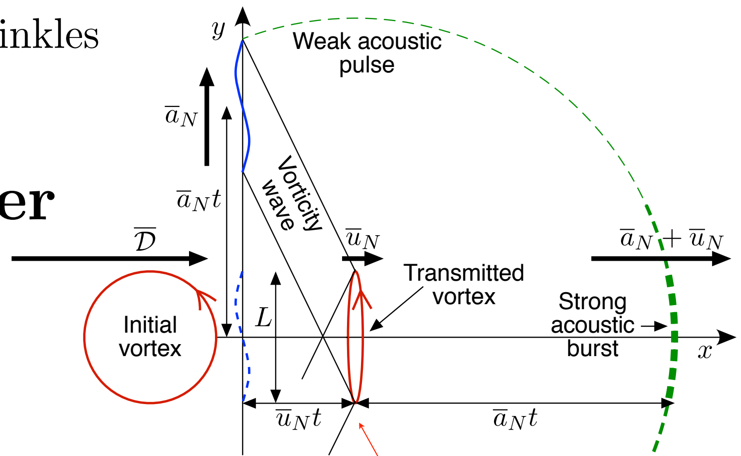
The disturbances of the front during the crossover can be described by a linear analysis

Interaction time $\tau_{int} = L/\overline{\mathcal{D}} \ll$ propagation time in the transverse direction of the wrinkles$L/\overline{a}_N$

After the interaction time, $t > \tau_{int}$, the wrinkled shock front propagates in a quiescent medium

2 timescales: short crossover and longer transverse propagation of the wrinkles

the crossover provides the initial conditions




squeezed vortex by compression

$\overline{u}_N \ll \overline{\mathcal{D}}$

## Linear analysis of the crossover

Similar analysis but with an upstream flow

$$\delta u_1(\mathbf{r}, t), \quad \delta w_1(\mathbf{r}, t), \quad \delta p_1(\mathbf{r}, t)$$

Rankine-Hugoniot (generalization of the relations p. 6)

(the subscript f denotes the value at the shock front of the upstream flow)

$$\frac{\delta p_N}{\overline{p}_N} - \frac{\delta p_{1f}}{\overline{p}_u} \approx 2\frac{(\delta u_{1f} - \dot{\alpha}_t)}{\overline{\mathcal{D}}}, \qquad \frac{\delta \rho_N}{\overline{\rho}_N} - \frac{\delta \rho_{1f}}{\overline{\rho}_u} = 2\left(\frac{\overline{a}_u}{\overline{a}_N}\right)^2 \frac{(\delta u_{1f} - \dot{\alpha}_t)}{\overline{\mathcal{D}}},$$

$$(\delta u_N - \dot{\alpha}_t) = \frac{[(\gamma-1)\overline{M}_u^2 - 2]}{2\overline{M}_u^2}(\delta u_{1f} - \dot{\alpha}_t), \quad \delta w_N \approx \overline{\mathcal{D}}\alpha'_y + \delta w_{1f},$$

$$\delta u_{1f}(y,t) = u_e|_{x=-\overline{\mathcal{D}}t}, \quad \delta w_{1f}(y,t) = w_e|_{x=-\overline{\mathcal{D}}t}, \quad \delta p_{1f}(y,t) = p_e|_{x=-\overline{\mathcal{D}}t}$$

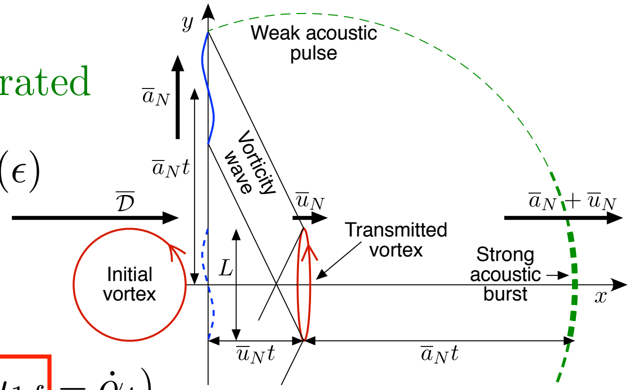
vortex

## *Acoustic burst*

Acoustic in the shocked gases (Doppler neglected for simplicity) $0 < t < \tau_{int}$

$$\frac{\partial u^{(a)}}{\partial t} \approx -\frac{1}{\bar\rho_N}\frac{\partial p}{\partial x}, \qquad \frac{\partial w^{(a)}}{\partial t} \approx -\frac{1}{\bar\rho_N}\frac{\partial p}{\partial y}, \qquad \frac{\partial^2 p}{\partial t^2} \approx \bar a_N^2\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)$$

$$x = 0, \quad 0 < t < \tau_{int}: \qquad \partial/\partial t = O(\overline{\mathcal{D}}/L), \qquad \partial/\partial y = O(1/L)$$




A quasi-planar and longitudinal pressure burst of transverse extension $L$ is generated

$$\partial p/\partial x \approx (1/\bar a_N)\partial p/\partial t \approx (\overline{\mathcal{D}}/\bar a_N)\delta p/L \approx \epsilon^{-1}(\partial p/\partial y) \Rightarrow |\delta w^{(a)}|/|\delta u^{(a)}| = O(\epsilon)$$

Rankine-Hugoniot $\quad \delta p_{1f}/\bar p_u$ is negligible $\quad \dfrac{\delta p_{1f}/\bar p_u}{v_e/\overline{\mathcal{D}}} \approx \dfrac{(v_e/\bar a_u)^2}{v_e/\overline{\mathcal{D}}} \approx \dfrac{(v_e/\bar a_u)}{\epsilon} \ll 1$

$$v_e/\bar a_u \ll \epsilon: \qquad \delta p_N/\bar p_N \approx 2(\delta u_{1f} - \dot\alpha_t)/\overline{\mathcal{D}}$$

$\delta u^{(a)} \approx \delta p/(\bar\rho_N \bar a_N)$

$$v_e/\bar a_u \ll \epsilon,\ 0 < t < \tau_{int}: \quad \delta u^{(a)}|_{x=0} \equiv \delta u_N^{(a)} \approx 2\boxed{(\bar a_N/\overline{\mathcal{D}})(\delta u_{1f}}-\dot\alpha_t)$$

## *Vorticity wave (transmitted vortex)*

$\delta u_N \approx \dot\alpha_t$

$$\delta u^{(i)}|_{x=0} \equiv \delta u_N^{(i)} = \delta u_N - \delta u_N^{(a)} \approx \dot\alpha_t - 2\boxed{(\bar a_N/\overline{\mathcal{D}})\delta u_{1f}}$$

$\delta w_N \approx \overline{\mathcal{D}}\alpha'_y + \delta w_{1f}$

$$\delta w^{(i)}|_{x=0} \equiv \delta w_N^{(i)} = \delta w_N - \delta w_N^{(a)} \approx \overline{\mathcal{D}}\alpha'_y + \delta w_{1f} - \delta w_N^{(a)}$$

Vorticity wave $\quad \delta u^{(i)} = \delta u_N^{(i)}(y, t - x/\bar u_N), \quad \delta w^{(i)} = \delta w_N^{(i)}(y, t - x/\bar u_N)$

$$\frac{\partial \delta u^{(i)}}{\partial x} = -\frac{1}{\bar u_N}\frac{\partial \delta u^{(i)}}{\partial t} = O\left(\frac{\overline{\mathcal{D}}}{\bar u_N}\frac{\delta u^{(i)}}{L}\right) = \boxed{O\left(\frac{\delta u^{(i)}}{\epsilon^2 L}\right)}$$

Incompressibility $\quad \partial\delta u^{(i)}/\delta x + \partial \delta w^{(i)}/\delta y = 0 \qquad |\delta u^{(i)}|/|\delta w^{(i)}| = O(\epsilon^2)$ The vorticity wave is quasi-parallel to the front

To leading order

$$\boxed{(\bar a_N/\overline{\mathcal{D}})\delta u_{1f}} = O(\epsilon v_e) \Rightarrow (\bar a_N/\overline{\mathcal{D}})\boxed{\partial \delta u_{1f}/\partial x} \approx v_e/(\epsilon L) \gg \boxed{\partial \delta w_{1f}/\partial y} = O(v_e/L)$$

$$\boxed{v_e/\bar a_u \ll \epsilon,\ 0 < t < \tau_{int}: \quad \dot\alpha_t \approx 2(\bar a_N/\overline{\mathcal{D}})\delta u_{1f}, \quad \alpha(y,t) = 2\frac{\bar a_N}{\overline{\mathcal{D}}}\int_{-\overline{\mathcal{D}}t}^{0} \mathrm{d}x\, \frac{u_e(x,y)}{\overline{\mathcal{D}}}}$$

longitudinal component of the vortex velocity

Wrinkles of very small amplitude are left on the shock front by the vortex

$$|\alpha| = O(\epsilon^2 L v_e/\bar a_u), \qquad |\alpha'_y| = O(\epsilon^2 v_e/\bar a_u)$$

# Shock-turbulence interaction

Clavin (2013) *J. Fluid Mech.*, **721**, 324-339

Strong shock propagating in a weakly turbulent flow

## Composite solution for a single vortex

*During crossover*

$$v_e/\bar{a}_u \ll \epsilon,\ 0<t<\tau_{int}: \quad \dot{\alpha}_t \approx 2(\bar{a}_N/\overline{\mathcal{D}})\delta u_{1f}, \quad \alpha(y,t) = 2\frac{\bar{a}_N}{\overline{\mathcal{D}}}\int_{-\overline{\mathcal{D}}t}^{0}\mathrm{d}x\frac{u_e(x,y)}{\overline{\mathcal{D}}}$$

beginning of interaction $< t <$ end of interaction : $\dfrac{\partial^2\alpha}{\partial t^2} \approx 2\dfrac{\bar{a}_N}{\overline{\mathcal{D}}}\dfrac{\partial \delta u_{1f}}{\partial t}$

valid during a short lapse of time of order $L/\overline{\mathcal{D}}$

*After crossover*

$$\frac{\partial^2\alpha}{\partial t^2} - \bar{a}_N^2\frac{\partial^2\alpha}{\partial y^2} + \overline{\mathcal{D}}\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial y}\right)^2 \approx 0$$

involves a time scale of evolution $L/\bar{a}_N$ longer than $L/\overline{\mathcal{D}}$

$\epsilon = \bar{a}_N/\overline{\mathcal{D}} \ll 1$

*Composite equation*

$$\frac{\partial^2\alpha}{\partial t^2} - \bar{a}_N^2\frac{\partial^2\alpha}{\partial y^2} + \overline{\mathcal{D}}\frac{\partial}{\partial t}\left(\frac{\partial\alpha}{\partial y}\right)^2 = 2\frac{\bar{a}_N}{\overline{\mathcal{D}}}\frac{\partial \delta u_{1f}}{\partial t}$$

frozen velocity field $u_e(x,y)$

$\delta u_{1f}(y,t) = u_e|_{x=-\overline{\mathcal{D}}t}$

short living forcing term

taking advantage of the two different time scales

# Model equation

*Extension to 3 dimensions*

$$\times\left(\frac{L\overline{\mathcal{D}}}{\overline{a}_N^3}\right) \qquad \frac{\partial^2\alpha}{\partial t^2} - \overline{a}_N^2\Delta\alpha + \overline{\mathcal{D}}\frac{\partial|\nabla\alpha|^2}{\partial t} = 2\frac{\overline{a}_N}{\overline{\mathcal{D}}}\frac{\partial\delta u_{1f}}{\partial t}$$

where $\delta u_{1f}(y,z,t) = u_e(x,y,z,t)|_{x=-\overline{\mathcal{D}}t}$

forcing term varying on the on the length scale $L$ and on time scale $L/\overline{\mathcal{D}}$

non-dimensional form

$$\eta \equiv y/L, \qquad \zeta \equiv z/L, \qquad \tau \equiv \overline{a}_N t/L, \qquad \phi \equiv \alpha/(\epsilon L) \qquad \epsilon \equiv \overline{a}_N/\overline{\mathcal{D}}$$

$$\boxed{\frac{\partial^2\phi}{\partial\tau^2} - \Delta\phi + \frac{\partial|\nabla\phi|^2}{\partial\tau} = \frac{\partial\psi(\eta,\zeta,\tau/\epsilon)}{\partial\tau}} \quad \text{where} \quad \psi \equiv 2\left(\frac{\delta u_{1f}}{\overline{a}_N}\right) \qquad |\psi| = O(v_e/\overline{a}_N)$$

$\psi$ is a small term varying on the short (reduced) time scale $\epsilon$ and on the (reduced) length scale unity

$\partial\psi/\partial\tau$ is a small (reduced) forcing term fluctuating rapidly $\quad |\partial\psi/\partial\tau| = O(v_e/(\epsilon\overline{a}_u)) \quad (v_e/\overline{a}_u \ll \epsilon)$

*Numerical results*

B. Denet *et al.* (2015) *Combust. Sci. Technol.* **187**, (1-2), 296-323

The characteristic cell size of the patterns

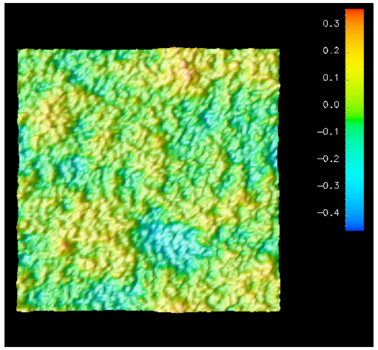
at the shock front is much larger

than the integral scale of the turbulence

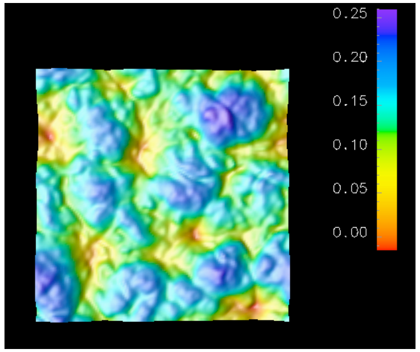
The size of the patterns looks to grow with time

Saturation by the box size ?

length scale of
the turbulence at the shock front




wrinkles of the shock front
at time $\tau = 4$ after starting the interaction



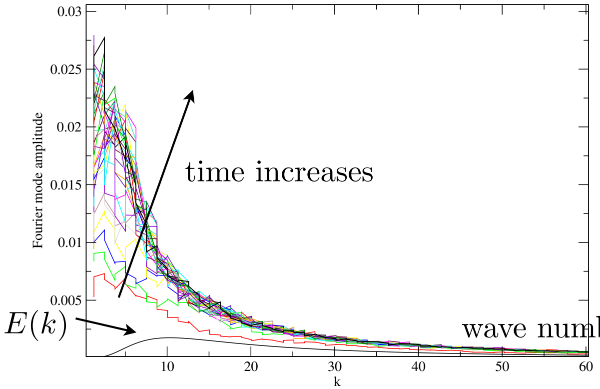
# Numerical analysis of the model equation

B. Denet *et al.* (2015) *Combust. Sci. Technol.* **187**, (1-2), 296-323

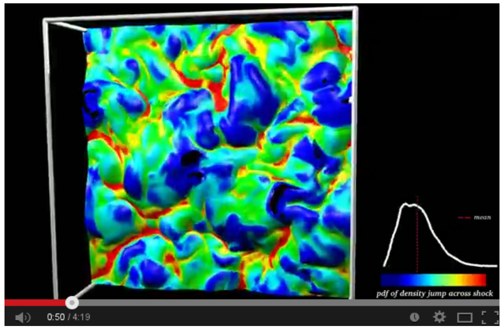
Spectral analysis of the pattern size

Evolution of the spectra of the wrinkles of the front shock

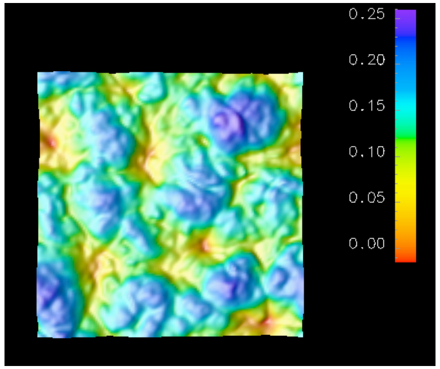
The size of the pattern increases with time




## Comparison with DNS

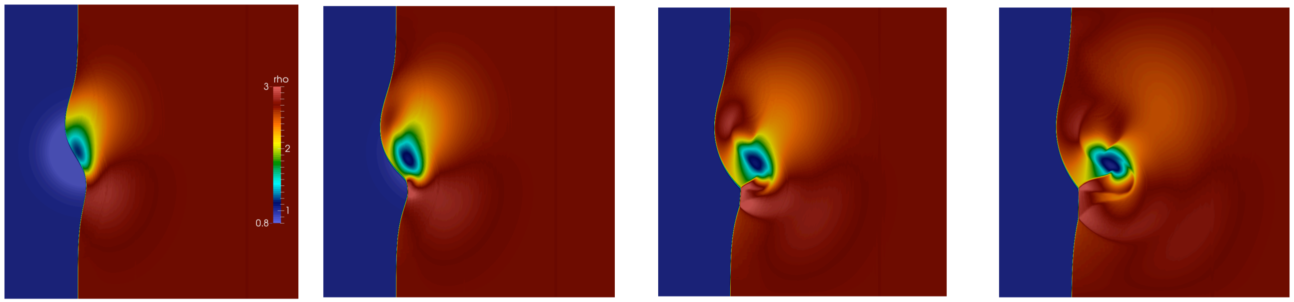


**DNS:** J. Larsson *et al.* (2013) *J. Fluid. Mech.* **717**, 293-321




**Model equation:** B. Denet *et al.* (2015)

DNS shock-vortex interaction B. Denet *et al.* (2015)



$$\overline{\mathcal{D}}/a_u = 2, \qquad v_e/a_u = 0.8, \qquad \gamma = 1.4$$

Two Mach stems are observed as in the model equation

*2025 Tsinghua Summer School*
*July 06 - 12, 2025*

# Dynamics of Combustion Waves in Premixed Gases

**Paul Clavin**

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**I**nstitut de **R**echerche sur les **P**hénomènes **H**ors **É**quilibre
ECM & CNRS

## Lecture XV
## Cellular detonations

# Lecture 15 : Cellular detonations

## 15-1. Cellular detonations at strong overdrive

*Order of magnitude. Scaling*

*Formulation*

*Outer flow in the burnt gas*

*Inner structure*

*Matching*

*Linear growth rate*

*Weakly nonlinear analysis*

## 15-2. Cellular instability near the CJ condition

*Formulation*

*Scaling*

*Model for CJ or near CJ regimes*

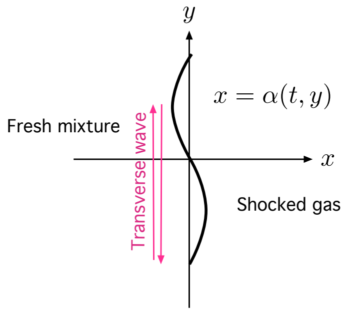
*Multidimensional stability analysis*

# Cellular detonations

## *Underlying mechanisms*

longitudinal oscillation of the complex shock reaction zone
$+$
transverse oscillatory mode of the lead shock

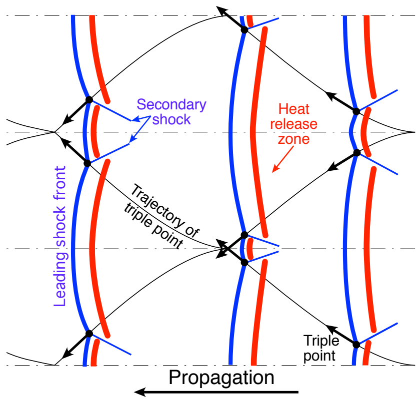
Longitudinal oscillation



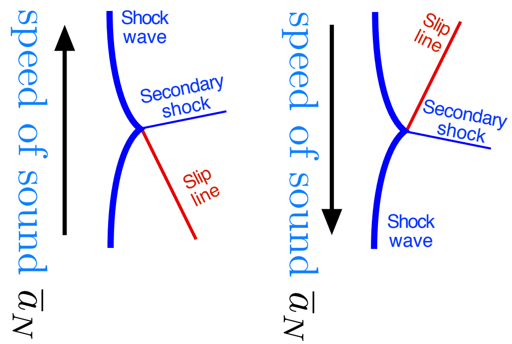

Pulsation $\approx 2\pi/\bar{t}_N$

trajectory of triple points

trajectory of Mach stems $=$ makings



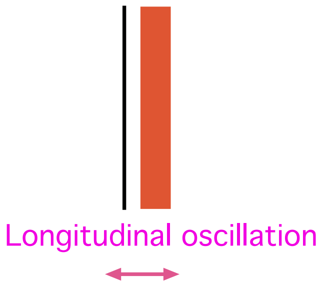

Mach stems propagating
in the transverse direction

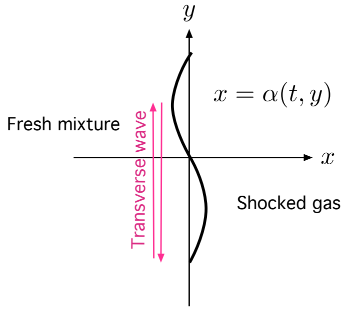


# Cellular detonations

*Underlying linear mechanisms*

longitudinal oscillation of the complex shock reaction zone
$+$
transverse oscillatory mode of the lead shock




# Cellular detonations at strong overdrive

Clavin(2002)(2017) Clavin et al (1997) Clavin Denet (2002) Daou Clavin (2003)
See the review by Clavin (2017) *Combust. Sci . Technol.* **189** (5) pp 747-775

*Order of magnitude and scaling*

Overdriven detonations in the Newtonian approximation $M_u \gg 1, \;\; M_N \ll 1$

$\overline{M}_N = \overline{u}_N / \overline{a}_N$

$$\boxed{\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \Leftrightarrow \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad\qquad (\gamma - 1) = O(\epsilon^2)}$$

$$\left.\begin{array}{r} \text{period of oscillation: } \bar{t}_N \equiv \tau_r(\overline{T}_N) \\ \text{transverse velocity of the shock disturbances: } \overline{a}_N \end{array}\right\}$$

$$\Rightarrow \qquad \boxed{\text{wavelength of unstable disturbance } \overline{a}_N \bar{t}_N = d_N/\epsilon}$$

detonation thickness $d_N = \overline{u}_N \bar{t}_N$ $\qquad \bar{t}_N \equiv \tau_r(\overline{T}_N)$

The unstable wavelengths are much larger than the detonation thickness by an order $1/\epsilon$

# Weakly unstable detonations at strong overdrive

Overdriven detonations in the Newtonian approximation

$\overline{M}_N = \overline{u}_N/\overline{a}_N$

$$\boxed{\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \quad \Leftrightarrow \quad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma - 1) = O(\epsilon^2)}$$

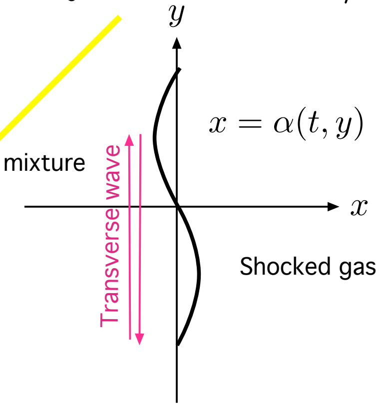
*Stability limit* $\boxed{q_N = O(\epsilon^2)}$ $\quad q_N \equiv Q/c_p\overline{T}_N$

The detonation is stable against the longitudinal disturbances

The unstable wavelengths are much larger than the detonation thickness by an order $1/\epsilon$

*Non-dimensional variables of order unity*

$$u \equiv \hat{u}/\overline{u}_N, \quad \boxed{\mathrm{v} \equiv \epsilon\hat{\mathrm{v}}/\overline{u}_N}, \quad p \equiv \hat{p}/\overline{p}_N, \quad \alpha \equiv \hat{\alpha}/d_N$$




$$\mathrm{x} \equiv \frac{1}{\overline{\rho}_N d_N}\int_{\hat{\alpha}}^{\hat{x}} \rho(\hat{x}', \hat{y}, \hat{t})\mathrm{d}\hat{x}', \quad \mathrm{t} \equiv \hat{t}/\overline{t}_N, \quad \boxed{\mathrm{y} \equiv \epsilon\hat{y}/d_N}$$

mass weighted coordinate

linear approximation

$$\bar{t}_N\frac{\mathrm{D}}{\mathrm{D}\hat{t}} = \frac{\partial}{\partial \mathrm{t}} + [\mathrm{m(t)} - v(\mathrm{x,y,t})]\frac{\partial}{\partial \mathrm{x}} + \cancel{\mathrm{v}\frac{\partial}{\partial \mathrm{y}}}$$

$$\mathrm{m(t)} \equiv 1 - \epsilon^2\partial\alpha/\partial\mathrm{t} = 1 + O(\epsilon^2) \qquad v(\mathrm{x,y,t}) \equiv \int_0^{\mathrm{x}} (\partial\mathrm{v}/\partial\mathrm{y})\mathrm{dx}' = O(1)$$

## Linearization

$$q_N = \epsilon^2 q_2 \qquad u = 1 + \epsilon^2 \overline{u}_2(\mathrm{x}) + \delta u \qquad T = 1 + \epsilon^2 \overline{T}_2(\mathrm{x}) + \delta T \qquad p = 1 + \epsilon^4 \overline{p}_4 + \delta p$$

$$\mathrm{v} = \delta v_0^{(i)} + O(\epsilon^2) \qquad \delta u = \delta u_0^{(i)} + O(\epsilon^2) \qquad \delta T = O(\epsilon^2) \qquad \delta p = O(\epsilon^2)$$

## Linear equations valid up to $\epsilon^4$

$\overline{\rho}_N \overline{u}_N^2 / \overline{p}_N$

Euler eq.

$$\epsilon^2 \left[\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u - v\frac{\mathrm{d}\overline{u}}{\mathrm{dx}}\right] = -\frac{\partial \delta p}{\partial \mathrm{x}}$$

$$\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\frac{\partial \mathrm{v}}{\partial \mathrm{y}} = -\overline{u}\frac{\partial^2 \delta p}{\partial \mathrm{y}^2}$$

Energy eq.

$\frac{1}{\gamma p}\frac{Dp}{Dt} + \nabla.\mathrm{u} = \frac{q}{c_p T}\dot{W}$

$$\frac{1}{\gamma}\frac{\overline{u}}{\overline{p}}\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta p + \underbrace{\frac{\partial}{\partial \mathrm{x}}\delta u + \overline{u}\frac{\partial}{\partial \mathrm{y}}\mathrm{v}}_{\text{divergence}} = q\delta w$$

rate of heat release

## Boundary conditions

Rankine-Hugoniot $\quad \mathrm{x} = 0:$

$$\delta u = \left[1 + (1/\overline{M}_u^2) - (\gamma - 1)/2\right]\partial\alpha/\partial \mathrm{t}, \qquad \mathrm{v} = \left[1 - (1/\overline{M}_u^2)\right]\partial\alpha/\partial \mathrm{y}$$

$$\delta p = -2\epsilon^2 \partial\alpha/\partial \mathrm{t}, \qquad \delta T_N/\overline{T}_N \approx -(\gamma - 1)\partial\alpha/\partial \mathrm{t}$$

$\mathrm{x} \to \infty:$ boundedness condition

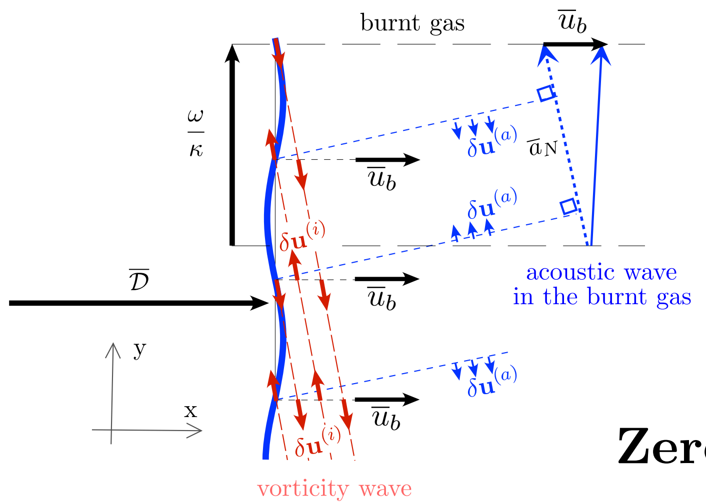
# Linear analysis in normal modes

$$x = \alpha(\mathbf{y}, \mathrm{t})$$

wrinkled front of the lead shock

$$\alpha = \tilde{\alpha}\exp(\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}) \qquad \delta f(\mathrm{x}, \mathbf{y}, \mathrm{t}) = \tilde{f}(\mathrm{x})\tilde{\alpha}\exp(\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y})$$

$$\sigma \equiv \hat{\sigma}\bar{t}_N, \qquad \kappa \equiv |\hat{k}|\overline{u}_N\bar{t}_N/\epsilon, \qquad \sigma(\kappa) = s(\kappa) \pm \mathrm{i}\omega(\kappa)$$




## Expansion in powers of $\epsilon$

$$\mathrm{v} = \delta v_0^{(i)} + O(\epsilon^2) \qquad \delta u = \delta u_0^{(i)} + O(\epsilon^2)$$

$$\boxed{\sigma(\kappa) = \sigma_0(\kappa) + \epsilon^2\sigma_2(\kappa) + ... \quad ?}$$

## Zeroth-order of the vorticity wave $\delta p^{(i)} = 0$

$$\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u_o^{(i)} = 0, \qquad \left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta v_o^{(i)} = 0$$

length and time scales of order unity

*Rankine-Hugoniot*

$$\mathrm{x} = 0: \quad \delta u_0^{(i)} = \partial\alpha/\partial \mathrm{t}, \quad \delta v_0^{(i)} = \partial\alpha/\partial \mathrm{y} \quad \Rightarrow \quad \delta u_0^{(i)} = \frac{\partial}{\partial \mathrm{t}}\alpha(\mathrm{t} - \mathrm{x}, \mathrm{y}), \quad \delta v_0^{(i)} = \frac{\partial}{\partial \mathrm{y}}\alpha(\mathrm{t} - \mathrm{x}, \mathrm{y})$$

*continuity*

$$\partial\delta u_0^{(i)}/\partial \mathrm{t} + \partial\delta v_0^{(i)}/\partial \mathrm{y} = 0 \quad \Rightarrow \quad \partial^2\alpha/\partial \mathrm{t}^2 - \partial^2\alpha/\partial \mathrm{y}^2 = 0$$

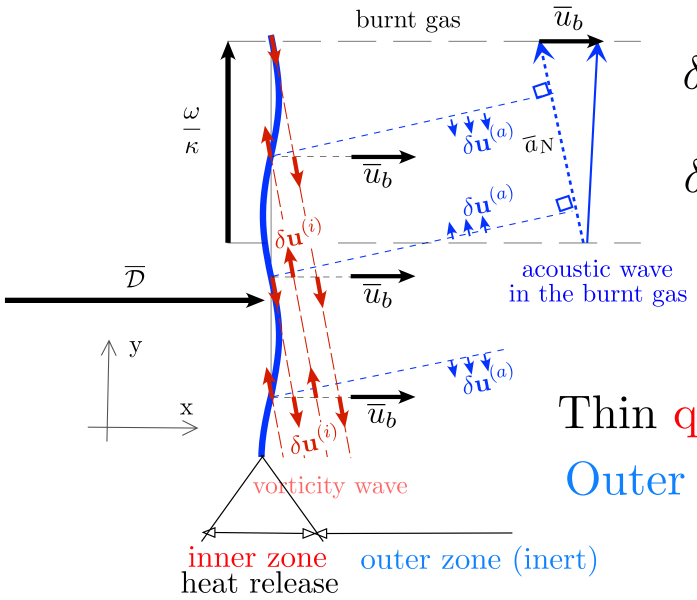
$$\boxed{\sigma_0 = \pm i\kappa}$$

transverse travelling waves

the growth or damping rate is of a smaller order

# Linear analysis at strong overdrive: acoustics +entropy-vortivity waves

R. Daou and P. Clavin (2003), *J. Fluid Mech.*, **482**, 181-206




$$\delta u = \delta u_0^{(i)}(\mathrm{x},\mathrm{t}) + \epsilon^2\left[\delta u_2^{(i)}(\mathrm{x},\mathrm{t}) + \delta u_2^{(a)}\right] + \ldots \qquad \delta p^{(i)} = 0$$

$$\delta \mathrm{v} = \delta \mathrm{v}_0^{(i)}(\mathrm{x},\mathrm{t}) + \epsilon^2\left[\delta \mathrm{v}_2^{(i)}(\mathrm{x},\mathrm{t}) + \delta \mathrm{v}_2^{(a)}\right] + \ldots \qquad \delta p = \epsilon^2 \delta p_2^{(a)} + \ldots$$

*Two zones :*

Thin quasi-isobaric inner layer where the heat is released $\mathrm{x} = O(1)$

Outer inert zone

*Acoustic waves in the outer zone*

$$\frac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p - \overline{a}_N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta p = 0 \quad \Rightarrow \quad \left\{\frac{1}{\gamma}\frac{\overline{u}_b}{\overline{p}_b}\left(\frac{\mathrm{d}}{\mathrm{dx}} + \sigma\right)^2 - \frac{1}{\epsilon^2}\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \overline{u}_b^2\kappa^2\right\}\tilde{p}(\mathrm{x}) = 0$$

$$\delta p = \tilde{p}(\mathrm{x})\mathrm{e}^{\sigma t + i\kappa . y} \qquad \boxed{\tilde{p}(\mathrm{x}) = \tilde{p}_b \exp(il\mathrm{x})}$$

$$\sigma(\kappa) = \sigma_0(\kappa) + \epsilon^2\sigma_2(\kappa)$$
$$\sigma_0(\kappa) = \pm i\kappa$$

$$\boxed{\sigma^2 + \kappa^2 = O(\epsilon^2)} \quad \Rightarrow \quad \mathrm{i}l = O(\varepsilon^2), \; \boxed{\mathrm{i}l = \epsilon^2 \mathrm{i}l_2,} \quad \mathrm{i}l_2 = O(1)$$

the sound waves propagate in the burnt gas in a direction quasi-parallel to the front

Rankine-Hugoniot

$$\mathrm{x} = 0: \quad \delta p = -2\epsilon^2\dot{\alpha}_\mathrm{t} \quad \Rightarrow \quad \boxed{\tilde{p}(\epsilon^2\mathrm{x}) = -2\epsilon^2\sigma\exp(\mathrm{i}\epsilon^2 l_2\mathrm{x}) + O(\epsilon^4)}$$

the sound waves are smaller than the entropy-vorticity wave by a factor $\epsilon^2$

# Outer inert zone

$$\sigma_0 = \pm i\kappa$$

$$\tilde{p}^{(a)}(\epsilon^2 \mathrm{x}) = -2\epsilon^2\sigma \exp(i\epsilon^2 l_2 \mathrm{x}) + O(\epsilon^4)$$

$$\tilde{u}_2^{(a)}(\epsilon^2 \mathrm{x}) = 2il_2 \exp(i\epsilon^2 l_2 \mathrm{x})$$

$$\widetilde{\partial v_2^{(a)}/\partial \mathrm{y}} = -2\kappa^2 \exp(i\epsilon^2 l_2 \mathrm{x})$$

$$\boxed{il_2 = \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\kappa^2}}$$

where $q_N \equiv \epsilon^2 q_2 \quad (\gamma - 1) \equiv \epsilon^2 h$

$$\tilde{u}^{(i)} = (\sigma_0 + \epsilon^2 u^*) \exp(-\sigma \mathrm{x})$$

$$\widetilde{\partial v^{(i)}/\partial y} = (-\kappa^2 + \epsilon^2 \sigma u^*) \exp(-\sigma \mathrm{x})$$

$u^* =$ contant of integration

$u^*$ and $\sigma_2$ are obtained by matching with the inner solution

# Inner structure of the detonation

$\tilde{U}^{(i)}(\mathrm{x}) = O(1) \qquad \tilde{\mathbf{V}}^{(i)}(\mathrm{x}) = O(1)$

$$\boxed{\tilde{u} \equiv \tilde{U}^{(i)}(\mathrm{x}) + \tilde{u}^{(a)}(\epsilon^2 \mathrm{x}) \qquad \tilde{\mathbf{v}} \equiv \tilde{\mathbf{V}}^{(i)}(\mathrm{x}) + \tilde{\mathbf{v}}^{(a)}(\epsilon^2 \mathrm{x})}$$

method similar to that used for galloping detonations

$$\nabla.\tilde{\mathbf{V}}^{(i)} \approx \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\kappa^2 \mathrm{e}^{-\sigma \mathrm{x}}$$

$$\tilde{U}^{(i)}(\mathrm{x}) - \left[1 + \frac{1}{M_U^2} - \frac{\gamma - 1}{2}\right]\sigma + 2\epsilon^2 \mathrm{i} l_2 + \overline{u}(\mathrm{x}) \int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)} d\mathrm{x}' \approx q_N \int_0^{\mathrm{x}} \left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)} \overline{\dot{\mathrm{w}}}\right) d\mathrm{x}'$$

# Matching

P.Clavin (2002), *Nonlinear PDE's in condensed matter and active flows*, 49-97, Kluwer

See also the review by Clavin (2017) *Combust. Sci . Technol.* **189** (5) pp 747-775

$$\lim_{x\gg 1} \tilde{U}^{(i)}(x) = \tilde{u}^{(i)}(x)$$

$$\tilde{u}^{(i)} = \left(\pm i\kappa + \epsilon^2 u^*\right) \exp(\pm i\kappa x - \epsilon^2 \sigma_2 x)$$

fast oscillation $x = O(1)$

slowly damped $x = O(1/\epsilon)$

$$\text{constant terms of } \lim_{x\gg 1} \tilde{U}^{(i)}(x) = 0$$

$$\boxed{\frac{\sigma_0}{\kappa}\sqrt{2\frac{\sigma_0\sigma_2}{\kappa^2} + h + q_2 - 1} - \frac{\sigma_0\sigma_2}{\kappa^2} + 1 - \frac{3}{4}h = \frac{q_2}{2}\mathcal{S}^{(i)}(\kappa)}$$

Algebraic equation of second degree for $\sigma_2$

where $\qquad \sigma_0 = \pm i\kappa \qquad q_m/(c_p \overline{T}_N) = \epsilon^2 q_2 \qquad (\gamma - 1) \equiv \epsilon^2 h$

$$\mathcal{S}^{(i)}(\kappa) \equiv \beta_N(\gamma - 1) s_{\beta_N}^{(i)}(\kappa) + s_q^{(i)}(\kappa)$$

$$s_{\beta_N}^{(i)}(\kappa) \equiv \int_0^\infty \Omega'_N(x) e^{-i\kappa x} dx \qquad s_q^{(i)}(\kappa) \equiv \int_0^\infty (1 + i\kappa x)\overline{\Omega}(x) e^{-i\kappa x} dx$$

$\overline{\Omega}(x)$ is the distribution of the non-dimensional rate of heat release in the steady state

$\Omega'_N(x)$ is its derivative with respect to the Neumann temperature, denoting its sensitivity to the overdriven regime

# Linear growth rate

R. Daou and P. Clavin (2003), *J. Fluid Mech.*, **482**, 181-206

$$\boxed{2\frac{\mathrm{Re}(\sigma)/\kappa,}{q_N} = -\mathrm{Im}\left[\mathcal{S}^{(i)}(\kappa)\right] - \frac{\overline{M}_N}{\sqrt{q_N}}\mathcal{S}^{(a)}(\kappa)}$$

**quasi-isobaric instability mechanism**

$\mathrm{Im}\ \mathcal{S}^{(i)} < 0$

$$\mathcal{S}^{(i)}(\kappa) \equiv \beta_N(\gamma-1)s^{(i)}_{\beta_N}(\kappa) + s^{(i)}_q(\kappa)$$

sensitivity to $T_N$

$$s^{(i)}_{\beta_N}(\kappa) \equiv \int_0^{\infty} \Omega'_N(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

strong instability due to wrinkling

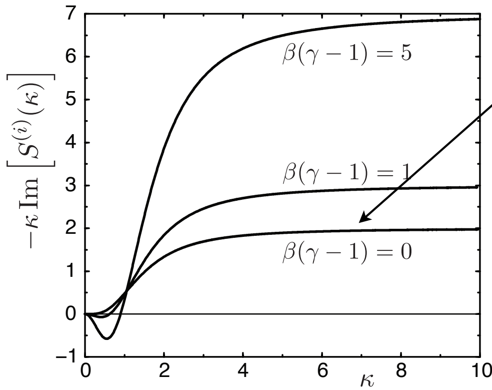
$$s^{(i)}_q(\kappa) \equiv \int_0^{\infty} (1+\mathrm{i}\kappa\mathrm{x})\overline{\Omega}(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

still working when $\beta_N = 0$

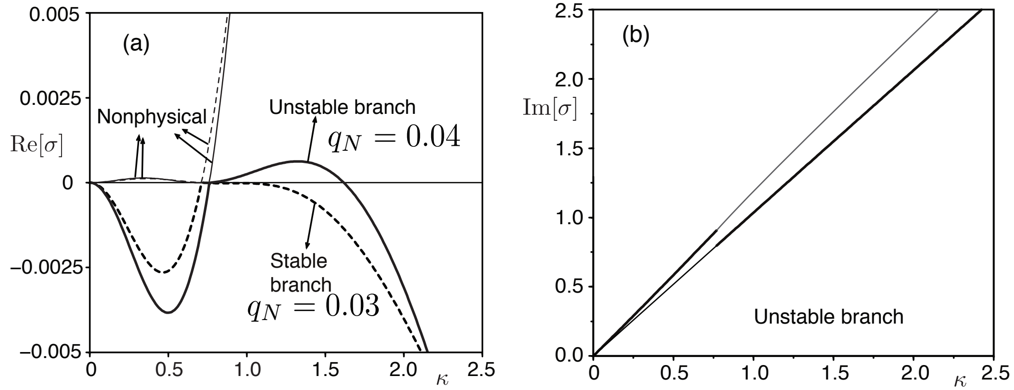
**stabilizing effect due to compressibility**

$$\mathcal{S}^{(a)}(\kappa) \equiv 2\left|\mathrm{Im}\sqrt{\frac{(\gamma-1)}{2q_N} + \mathcal{S}^{(i)}(\kappa) - 1}\right| > 0$$

$q_N \nearrow \;\Rightarrow \begin{cases} \text{stabilisation} \searrow \\ \text{instability} \nearrow \end{cases}$



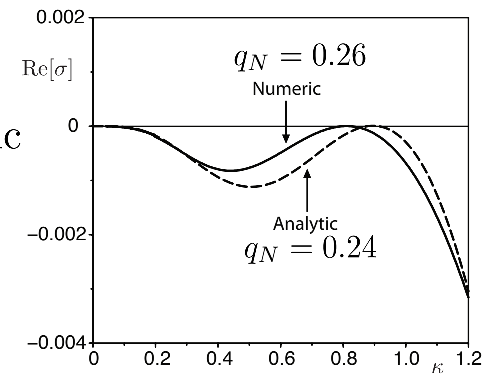

Arrhenius law with $\beta_N q_N = 0.1$




$(\gamma-1)=0.1,\ \beta_N=10,\ M_u^2=50$

**Threshold of linear instability for $\beta_N = 0$** $\gamma = 1.05,\ M_u^2 = 20$ $(M_N = 0.267)$

good agreement between theory and numeric



# Weakly nonlinear analysis of cellular detonations

P. Clavin and B. Denet (2002), Phys. *Rev. Lett.,* **88** *(4)* 044502

Near to the instability threshold the dominant nonlinear effects are those responsible for singularity formation on the inert shock front (representative of Mach stem), see p.12 of lecture XIV

## *Model equation*

A weakly nonlinear analysis leads to a combination of the linear equation for the multidimensional instability of an overdriven detonation and the nonlinear equation for the lead shock

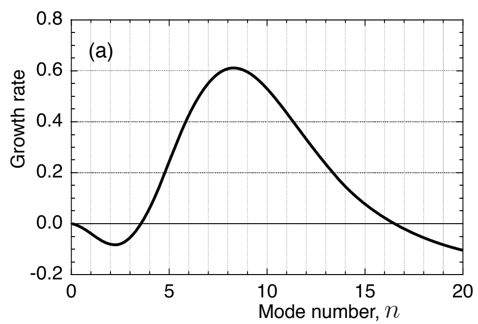
equation of the detonation front $x = \alpha(\mathbf{y}, \mathrm{t})$

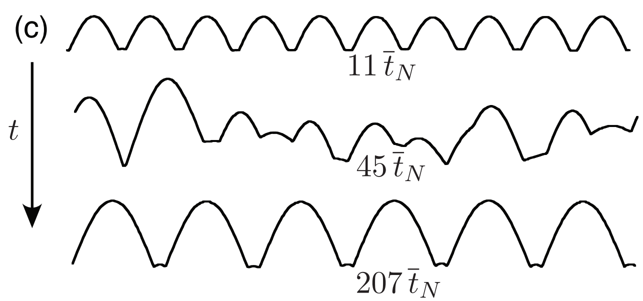
$$\boxed{\frac{\partial^2 \alpha}{\partial \mathrm{t}^2} - c^2 \nabla^2 \alpha + \frac{\partial |\nabla \alpha|^2}{\partial \mathrm{t}} = \underbrace{q_N L^{(i)}(\alpha)}_{\text{quasi-isobaric instability}} - 2\overline{M}_N \sqrt{q_N} \frac{\partial}{\partial \mathrm{t}} L^{(a)}(\alpha)}$$

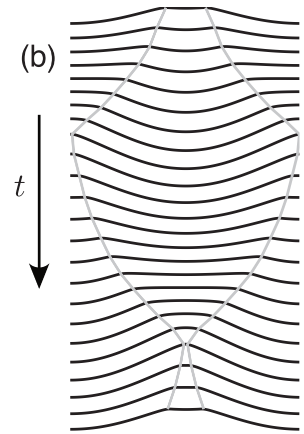
$c^2 = 1 + 3(\gamma - 1)/2 \qquad L^{(i)}(\phi) = \beta_N(\gamma - 1) l^{(i)}_{\beta_N}(\alpha) + l^{(i)}_q(\alpha) \qquad L^{(a)}(\alpha) \approx \kappa\tilde{\alpha}/2$ in Fourier space

$$l^{(i)}_{\beta_N}(\alpha) = \frac{\partial^2}{\partial \mathrm{t}^2} \int_0^\infty \Omega'_N(\mathrm{x}) \alpha(\mathrm{t} - \mathrm{x}) \mathrm{dx}, \quad l^{(i)}_q(\alpha) = \nabla^2 \int_0^\infty \Psi(\mathrm{x}) \alpha(\mathrm{t} - \mathrm{x})/\mathrm{dx} \quad \text{where } \Psi(\mathrm{x}) \equiv \overline{\Omega}(\mathrm{x}) + \mathrm{d}(\mathrm{x}\overline{\Omega})/\mathrm{dx}$$

Good qualitative agreement with the experimental observation



# Conclusion for large overdrives

nonlinear dynamics of the lead shock — linear terms due to combustion

$$\frac{\partial^2 \alpha}{\partial t^2} - \frac{\partial^2 \alpha}{\partial y^2} + \frac{\partial}{\partial t}\left(\frac{\partial \alpha}{\partial y}\right)^2 = [q_N G(\alpha) - M_N \sqrt{q_N} D(\alpha)]$$

NL term

linear growth (quasi-isobaric mechanism) — linear damping (compressible effect)

*Quasi-iobaric instability mechanism*

$$G(\alpha) = \beta_N(\gamma - 1)\frac{\partial^2}{\partial t^2}G_p(\alpha) + \frac{\partial^2}{\partial y^2}G_w(\alpha)$$

1-D galloping mechanism — modification due to wrinkling

$$G_p(\alpha) = \int_0^\infty \Omega'_N(\mathrm{x})\alpha(\mathrm{t}-\mathrm{x},\mathrm{y})\mathrm{dx}$$

$$G_w^(\alpha) = \int_0^\infty \Psi(\mathrm{x})\alpha(\mathrm{t}-\mathrm{x},\mathrm{y})/\mathrm{dx}$$

thermal sensitivity of the distribution of heat release rate

$$\Omega'_N(\mathrm{x}) \equiv \partial\Omega/\partial\Theta_N$$

$$\Psi(\mathrm{x}) \equiv \Omega(\mathrm{x}) + \mathrm{d}(\mathrm{x}\Omega)/\mathrm{dx}$$

Multdim instability before 1-D instability

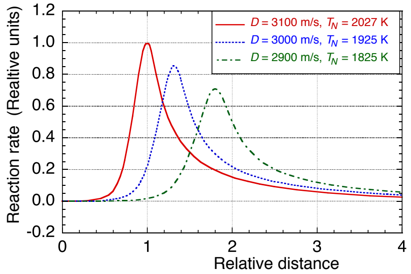
Superposition of two mechanisms

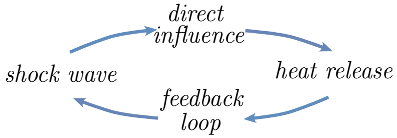
Nonlinear dynamics of the lead shock + linear oscillation due to the heat release

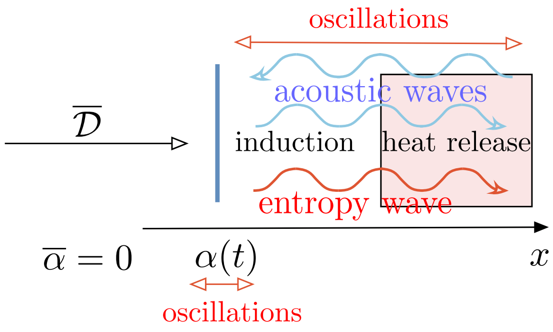
Oscillatory frequency $\omega \bar{t}_N = O(1)$ where $\bar{t}_N \equiv \tau_{reac}(\overline{T}_N)$

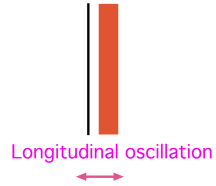
Nonlinear wavelength selection $\Lambda \approx a\,\overline{u}_N \bar{t}_N / \overline{M}_N$ $\quad a \in [2-5]$

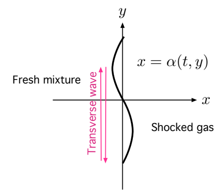
# Dynamics of the detonation waves near the CJ regime



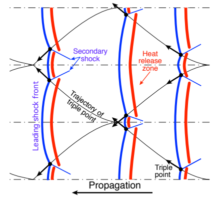




Two different coupling mechanisms: { acoustic waves / entropy wave




Analytical solutions have been obtained in **two limiting cases** (opposite conditions):

*Strongly overdriven detonation in the Newtonian limit*

P. Clavin and L. He (1996), *J. Fluid Mech.* **306,** *353-378*

quasi-isobaric flow

dominant mechanism: entropy wave

*CJ (or near CJ) close to the instability threshold*

P. Clavin and F.A. Williams (2009), *J. Fluid Mech.* **624,** 125-*150*

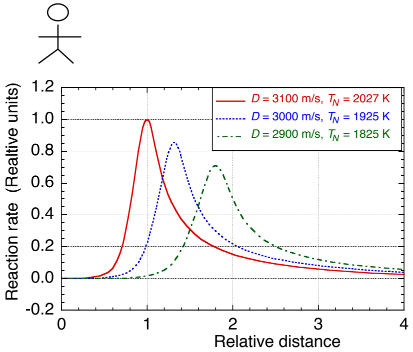
Transonic flow

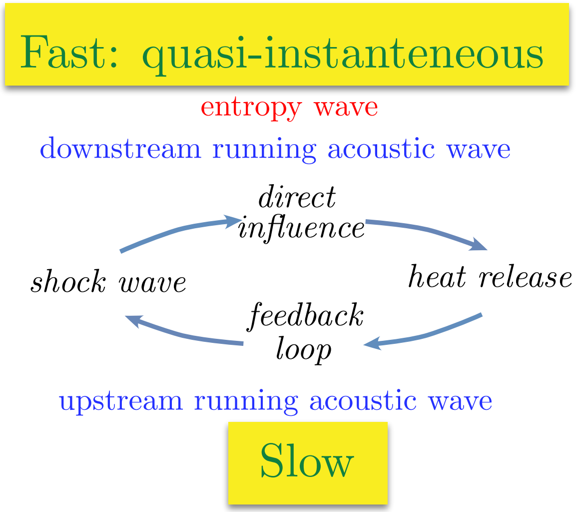
Slowest (dominant) mechanism in the loops: Upwards propagating acoustic wave

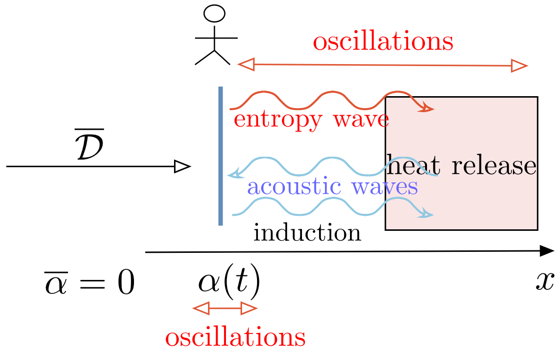
# Cellular instability near the CJ regime

## *( basic approximation: small heat release )*

Clavin Williams 2009, 2012




*Distinguished limit*

Near the CJ regime the instability threshold concerns transonic conditions associated with small heat release

Clavin Williams 2009

Same distinguished limit as in pp 9-10 lecture XII

⚠ notation

$$\epsilon^2 \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_p T_u} \ll 1 \qquad (\gamma - 1) = O(\epsilon)$$

# Cellular CJ detonation (small heat release)

Clavin Williams 2009, 2012

## Formulation of the free boundary problem

### *Reactive Euler equations in 2-D geometry*

$$\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + u\partial/\partial x + w\partial/\partial y \qquad \mathrm{D}^{\pm}/\mathrm{D}t = \mathrm{D}/\mathrm{D}t \pm a\,\partial/\partial x$$

$$\frac{1}{\gamma p}\frac{\mathrm{D}^{\pm}p}{\mathrm{D}t} \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}u}{\mathrm{D}t} = \frac{q_m}{c_p T}\frac{\dot{\mathrm{w}}}{\bar{t}_N} - \frac{\partial w}{\partial y} \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \cancel{\frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t}} = \frac{q_m}{c_p T_u}\frac{\dot{\mathrm{w}}}{\bar{t}_N} \qquad \frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

compressional heating

### *Boundary conditions*

On the lead shock $x = \alpha(y,t)$ : *Neumann conditions*

At infty in the burned gas: *boundedness condition*

### *Distinguished limit*

Transonic flow $\quad M \equiv a_u^{-1}\partial\alpha/\partial t$

High thermal sensitivity of the inductio

$$\boxed{\epsilon^2 \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_p T_u} \ll 1, \qquad M_{CJ} - 1 \approx \epsilon, \qquad (\gamma-1)/\epsilon < 1, \qquad \epsilon(\gamma-1)\frac{E}{k_B T_u} = O(1)}$$

notation $\qquad T_u$ !

$$\frac{E}{k_B T_N} \equiv \left[\frac{T}{\dot{\mathrm{w}}}\frac{\partial\dot{\mathrm{w}}}{\partial T}\right]_{T=T_N}$$

### *Scaling laws*

$$t/\bar{t}_N = \epsilon\tau, \quad x/a_u\bar{t}_N = \xi, \quad y/a_u\bar{t}_N = \eta/\epsilon^{1/2}$$

$$u/a_u = 1 + \epsilon\mu, \quad v/a_u = \epsilon^{3/2}\nu, \quad \ln(p/p_u) = \epsilon\pi, \quad (T - T_u)/T_u = \epsilon^2\theta$$

# Cellular CJ detonation (small heat release)

Clavin Williams 2009, 2012

$$\epsilon^2 \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_p T_u} \ll 1$$

*Normal mode analysis*

Equation for the front of the lead shock

$$\xi = \alpha(\eta,\tau) = \mathrm{e}^{\sigma\tau + i\kappa\eta}$$

$$t/\bar{t}_N = \epsilon\tau, \quad x/a_u\bar{t}_N = \xi, \quad y/a_u\bar{t}_N = \eta/\epsilon^{1/2}$$

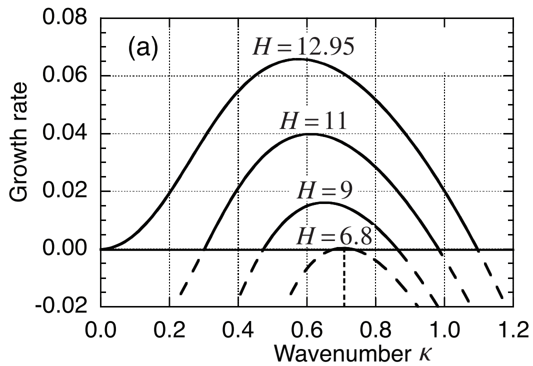
*Analytical result*

Model for the distribution of heat release rate of the CJ wave

$$\Omega(\xi) = \xi^n \mathrm{e}^{-\xi}/n!, \quad \xi \geq 0, \quad n \geq 1$$

Algebraic equation for $\sigma(\kappa)$ with a single parameter $\boxed{H = (n+1)(\gamma-1)\frac{q_m}{c_p T_N}\frac{E}{k_b T_N}}$

$$\boxed{4\left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)^{n+2} = H\sigma + \left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)}$$




Bifurcation scenario similar to that of the strongly overdriven regimes

**Generic character of the results:**

The bifurcation scenario of CJ waves is similar to that at large overdrive

The integral equations are of the same type

Increasing the thermal sensitivity of the induction length

or stiffening the distribution of heat release promote the instability.

Detonations are unstable against the transverse disturbances before the longitudinal ones

Unstable wavelengths are larger than the detonation thickness

Thank you for your attention

# Details of the calculation (strong overdrive)

## *Non-dimensional variables of order unity*

details of the analysis

denoting $\hat{w}$ the original dimensional quantities and $w$ the dimensionless quantity

$$u \equiv \hat{u}/\overline{u}_N, \ \mathbf{v} \equiv \epsilon\hat{\mathbf{v}}/\overline{u}_N, \ p \equiv \hat{p}/\overline{p}_N, \ T \equiv \hat{T}/\overline{T}_N \ \text{ and } \alpha \equiv \hat{\alpha}/d_N, \ d_N \equiv \overline{u}_N\overline{t}_N$$

notations

where the scaling of the transverse velocity $\hat{\mathbf{v}}$ comes from the Rankine-Hugoniot condition

$\hat{\mathbf{v}}_N/\overline{u}_N \propto (\partial\hat{\alpha}/\partial y, \ \partial\hat{\alpha}/\partial z)$ and the scaling of the transverse coordinates $\partial/\partial y = \epsilon d_N^{-1}\partial/\partial \mathrm{y}, \ \ \partial/\partial z = \epsilon d_N^{-1}\partial/\partial \mathrm{z}$

# Formulation (Clavin et al. 1997, Clavin 2002)

$$\mathrm{x} \equiv \frac{1}{\overline{\rho}_N\overline{u}_N\overline{t}_N}\int_{\alpha(\mathrm{y,z,t})}^{x}\rho(x',y,z,t)\mathrm{d}x'$$

$$\mathbf{y} \equiv (\epsilon\, y/d_N, \epsilon\, z/d_N) \qquad \mathrm{t} \equiv \frac{t}{\overline{t}_N}$$

$$\Rightarrow \boxed{\frac{\mathrm{D}}{\mathrm{Dt}} = \frac{\partial}{\partial \mathrm{t}} + [\mathrm{m(t)} - \upsilon(\mathrm{x},\mathbf{y},\mathrm{t})]\frac{\partial}{\partial \mathrm{x}} + \mathbf{v}.\nabla_{\mathbf{y}}}$$

where $\boxed{\mathrm{m(t)} \equiv 1 - (\partial\hat{\alpha}/\partial\hat{\mathrm{t}})/\overline{\mathcal{D}} \quad \text{and} \quad \upsilon(\mathrm{x},\mathbf{y},\mathrm{t}) \equiv \int_0^{\mathrm{x}} \nabla_{\mathbf{y}}.\mathbf{v}\,\mathrm{dx}' = O(1)}$

$$\frac{m(t)}{\sqrt{1+|\nabla\alpha|^2}}$$ mass flux across the leading shock

change of variable

$\mathbf{v}(\mathrm{t},\mathbf{y}) \qquad \nabla_{\mathbf{y}}.\mathbf{v} = O(1)$

continuity (in the linear approximation) $\Rightarrow \dfrac{\partial r}{\partial \mathrm{t}} + \mathrm{m(t)}\dfrac{\partial r}{\partial \mathrm{x}} = \dfrac{\partial}{\partial \mathrm{x}}[u + \overline{r}(\mathrm{x})\upsilon(\mathrm{x},\mathbf{y},\mathrm{t})]$ where $r(\mathrm{x},\mathbf{y},\mathrm{t}) \equiv \overline{\rho}_N/\hat{\rho}$

notations

## *Stability limit*

$q_N \equiv q_m/c_p\overline{T}_N$ $\qquad \boxed{q_N = O(\epsilon^2)}$

$\overline{u}_N/\overline{\mathcal{D}} = O(\epsilon^2) \ \Rightarrow \boxed{\mathrm{m(t)} = 1 + O(\epsilon^2)}$

Clavin Denet (2002) Daou Clavin (2003)

small variations across the shocked gas

## *Expansion in powers of $\epsilon^2$*

$$\boxed{q_N = \epsilon^2 q_2 \qquad u = 1 + \epsilon^2\overline{u}_2(\mathrm{x}) + \delta u \qquad T = 1 + \epsilon^2\overline{T}_2(\mathrm{x}) + \delta T \qquad p = 1 + \epsilon^4\overline{p}_4 + \delta p}$$

## *Linear equations*

$$\epsilon^2\left[\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u - \upsilon\frac{\mathrm{d}\overline{u}}{\mathrm{dx}}\right] = -\frac{\partial\delta p}{\partial \mathrm{x}} \qquad \left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)(\nabla.\mathbf{v}) = -\overline{u}\nabla^2\delta p$$

$\overline{\rho}_N\overline{u}_N^2/\overline{p}_N = \epsilon^2$

$$\frac{1}{\gamma}\frac{\overline{u}}{\overline{p}}\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta p + \frac{\partial}{\partial \mathrm{x}}(\delta u + \overline{u}\upsilon) = q_N(\delta\dot{\mathrm{w}} + \upsilon\overline{\dot{\mathrm{w}}}), \qquad q_N = \epsilon^2 q_2$$

## *Rankine-Hugoniot conditions*

$\dot{\alpha}_{\mathrm{t}} \equiv (\overline{t}_N/d_N)\partial\hat{\alpha}/\partial t = \overline{u}_N^{-1}\partial\hat{\alpha}/\partial t = O(1) \qquad \nabla\alpha \equiv \epsilon d_N^{-1}\left(\dfrac{\partial\hat{\alpha}}{\partial y}, \dfrac{\partial\hat{\alpha}}{\partial z}\right) = O(1)$

$$\mathrm{x} = 0: \quad \delta u \approx \left[1 + \frac{1}{M_u^2} - \frac{(\gamma-1)}{2}\right]\dot{\alpha}_{\mathrm{t}}, \quad \mathbf{v} \approx \left[1 - \frac{1}{M_u^2}\right]\nabla\alpha, \qquad \delta p \approx -2\epsilon^2\dot{\alpha}_{\mathrm{t}}, \quad \delta T_N \approx -(\gamma-1)\dot{\alpha}_{\mathrm{t}}$$

$\alpha(\mathbf{y},\mathrm{t}) = \tilde{\alpha}\mathrm{e}^{\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}}$ $\quad \delta u = \tilde{u}(\mathrm{x})\tilde{\alpha}\mathrm{e}^{\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}}$ $\quad \delta \mathbf{v} = \tilde{\mathbf{v}}(\mathrm{x})\tilde{\alpha}\mathrm{e}^{\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}}$ $\quad \delta p = \tilde{p}(\mathrm{x})\tilde{\alpha}\mathrm{e}^{\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}}$

## *Outer flow (burnt gas)*

details of the analysis

$$\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u^{(i)} = 0, \quad \left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\mathbf{v}^{(i)} = 0$$ valid up to $\epsilon^2$ in the burnt gas $\quad \boxed{\sigma_0 = \pm \mathrm{i}\kappa} \quad \boxed{\sigma_2 ?}$

$$\tilde{u}^{(i)} = \left[\sigma_0 + \epsilon^2 \tilde{u}^{(i)}_{b2}\right]\mathrm{e}^{-\sigma \mathrm{x}} \qquad \nabla.\tilde{\mathbf{v}}^{(i)} = \left[-\kappa^2 + \epsilon^2 \nabla.\tilde{\mathbf{v}}^{(i)}_{b2}\right]\mathrm{e}^{-\sigma \mathrm{x}}$$

$\nabla.\tilde{\mathbf{v}}^{(i)}_{b2} = \sigma_0 \tilde{u}^{(i)}_{b2}$

unknown constants of integration

$\tilde{p}(\epsilon^2 \mathrm{x}) = -2\epsilon^2 \sigma \mathrm{e}^{\mathrm{i}\epsilon^2 l_2 \mathrm{x}} \quad \Rightarrow \quad \tilde{u}^{(a)} = 2\epsilon^2 \mathrm{i} l_2 \mathrm{e}^{\mathrm{i}\epsilon^2 l_2 \mathrm{x}} \qquad \nabla.\tilde{\mathbf{v}}^{(a)} = -2\epsilon^2\kappa^2 \mathrm{e}^{\mathrm{i}\epsilon^2 l_2 \mathrm{x}}$

the acoustic flow is of order $\epsilon^2$

$$\tilde{p}(\epsilon^2 \mathrm{x}) = -2\epsilon^2 \sigma \exp(\mathrm{i}\epsilon^2 l_2 \mathrm{x}) + O(\epsilon^4) \qquad \mathrm{i}l_2 \approx \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\kappa^2}$$

the acoustic flow is small, of order $\epsilon^2$, and varies on a long length scale

# Inner detonation structure (inner zone)

## *Inner flow (reacting gas)*

splitting $\quad \boxed{\tilde{u} \equiv \tilde{U}^{(i)}(\mathrm{x}) + \tilde{u}^{(a)}(\epsilon^2 \mathrm{x}) \qquad \tilde{\mathbf{v}} \equiv \tilde{\mathbf{V}}^{(i)}(\mathrm{x}) + \tilde{\mathbf{v}}^{(a)}(\epsilon^2 \mathrm{x})}$ $\quad \tilde{U}^{(i)}(\mathrm{x}) = O(1) \quad \tilde{\mathbf{V}}^{(i)}(\mathrm{x}) = O(1)$

$$\frac{1}{\gamma}\frac{\overline{u}}{\overline{p}}\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta p + \frac{\partial}{\partial \mathrm{x}}(\delta u + \overline{u}v) = q_N(\delta\dot{\mathrm{w}} + v\overline{\dot{\mathrm{w}}}), \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{dx}}\left[\tilde{U}^{(i)} + \overline{u}(\mathrm{x})\tilde{v}^{(i)}(\mathrm{x})\right] \approx q_N\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}^{(i)}_0\overline{\dot{\mathrm{w}}}\right) \qquad \tilde{v}^{(i)} \equiv \int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}'$$

subtracting out the acoustics

valid up to $\epsilon^2$ $\quad q_N = \epsilon^2 q_2$

$$\mathrm{d}\overline{u}/\mathrm{dx} = q_N \overline{\dot{\mathrm{w}}} \quad \Rightarrow \quad \mathrm{d}\tilde{U}^{(i)}/\mathrm{dx} + \overline{u}\nabla.\tilde{\mathbf{V}}^{(i)} \approx q_N \tilde{\dot{\mathrm{w}}}$$

$$\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)(\nabla.\mathbf{v}) = -\overline{u}\nabla^2\delta p \quad \Rightarrow \quad (\partial/\partial\mathrm{t} + \partial/\partial\mathrm{x})\nabla.\mathbf{V}^{(i)} \approx 0$$ valid up to $\epsilon^2$

subtracting out the acoustics

$\mathrm{x} = 0: \quad \mathbf{v} \approx \left[1 - \frac{1}{M_u^2}\right]\nabla\alpha \qquad \nabla.\tilde{\mathbf{v}}^{(a)} = -2\epsilon^2\kappa^2 \quad \Rightarrow \quad \nabla.\tilde{\mathbf{V}}^{(i)} \approx \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\kappa^2 \mathrm{e}^{-\sigma \mathrm{x}}$ valid up to $\epsilon^2$

Rankine-Hugoniot see p.6 lecture X and p.6 lecture XIII

$\mathrm{x} = 0: \quad \delta u \approx \left[1 + \frac{1}{M_u^2} - \frac{(\gamma - 1)}{2}\right]\dot{\alpha}_{\mathrm{t}} \qquad \tilde{u}^{(a)} = 2\epsilon^2 \mathrm{i} l_2 \quad \Rightarrow \quad \tilde{U}^{(i)}(\mathrm{x}) - \left[1 + \frac{1}{M_U^2} - \frac{\gamma - 1}{2}\right]\sigma + 2\epsilon^2 \mathrm{i} l_2 + \overline{u}(\mathrm{x})\int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}' \approx q_N \int_0^{\mathrm{x}}\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}^{(i)}_0\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}'$

$\sigma_0 = \pm i\kappa$ $\quad \boxed{\sigma = \pm i\kappa + \epsilon^2\sigma_2^2}$ $\quad \boxed{\sigma_2 ?}$

details of the analysis

# Matching

**internal solution**

$$\tilde{U}^{(i)}(\mathrm{x}) - \left[1 + \frac{1}{M_U^2} - \frac{\gamma-1}{2}\right]\sigma + 2\epsilon^2 \mathrm{i}l_2 + \overline{u}(\mathrm{x}) \int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}' \approx q_N \int_0^{\mathrm{x}} \left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' \qquad q_N = \epsilon^2 q_2$$

$$\nabla.\tilde{\mathbf{V}}^{(i)} \approx \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\kappa^2 \mathrm{e}^{-\sigma\mathrm{x}} \quad \Rightarrow \quad \int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}' = \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\frac{\kappa^2}{\sigma}\left(1 - \mathrm{e}^{-\sigma\mathrm{x}}\right)$$

at the end of the reaction $\overline{w} = 0,\ \tilde{w} = 0 : \tilde{U}^{(i)}(\mathrm{x}) \rightarrow$ constant term + oscillatory term

constant term

$$\boxed{\left[1 + \frac{1}{M_u^2} - \frac{\gamma-1}{2}\right]\sigma - 2\epsilon^2 \mathrm{i}l_2 - \overline{u}_b\left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\frac{\kappa^2}{\sigma} + q_N \int_0^{\infty}\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' = 0}$$

$$\mathrm{i}l_2 \approx \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\kappa^2}$$

**external solution**

$$\tilde{u}^{(i)} = \left[\sigma_0 + \epsilon^2 \tilde{u}_{b2}^{(i)}\right]\mathrm{e}^{-\sigma\mathrm{x}}$$

oscillatory term with an amplitude varying on a long length scale, $\mathrm{Re}(\sigma) = O(\epsilon^2)$

matching $\Rightarrow$ the constant term of the internal solution should be zero $\Rightarrow$ $\boxed{\text{equation for } \sigma \text{ when } \tilde{\dot{\mathrm{w}}} \text{ is known}}$

*Reaction rate and dispersion relation* $\sigma_2(\kappa)$

$$\frac{\partial T}{\partial \mathrm{t}} + \frac{\partial T}{\partial \mathrm{x}} = q_N(1 + v_0^{(i)})\dot{\mathrm{w}} \qquad \frac{\partial \psi}{\partial \mathrm{t}} + \frac{\partial \psi}{\partial \mathrm{x}} = (1 + v_0^{(i)})\dot{\mathrm{w}} \qquad v_0^{(i)} = -\frac{\kappa^2}{\sigma_0}\left(1 - \mathrm{e}^{-\sigma_0\mathrm{x}}\right) \qquad \mathrm{x} = 0: \quad T = T_N(\mathbf{y}, t), \quad \psi = 0$$

method similar to that used for galloping detonations see p.7-8 lecture XII additional effect of wrinkling

Clavin et al (1997)
Daou Clavin (2003)

$$\int_0^{\infty}\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' = \frac{\kappa^2}{\sigma}\left[-1 + \mathcal{S}^{(i)}(\kappa)\right] \quad \Rightarrow \quad \boxed{\frac{\sigma_0}{\kappa}\sqrt{2\frac{\sigma_0\sigma_2}{\kappa^2} + h + q_2 - 1} - \frac{\sigma_0\sigma_2}{\kappa^2} + 1 - \frac{3}{4}h = \frac{q_2}{2}\mathcal{S}^{(i)}(\kappa)}$$

equation for $\sigma_2(\kappa)$

$$\mathcal{S}^{(i)}(\kappa) \equiv \beta_N(\gamma-1)s_{\beta_N}^{(i)}(\kappa) + s_q^{(i)}(\kappa) \qquad s_{\beta_N}^{(i)}(\kappa) \equiv \int_0^{\infty}\Omega_N'(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx} \qquad s_q^{(i)}(\kappa) \equiv \int_0^{\infty}(1 + \mathrm{i}\kappa\mathrm{x})\overline{\Omega}(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

# Details of the calculation (CJ wave)

# Cellular instability near the CJ condition

## *(small heat release)*

Clavin Williams 2009, 2012

## Formulation

Extension of the analysis of galloping detonations (planar case) pp 9-13 lecture XII

### *Reactive Euler equations in 2-D geometry*

Same as in p.9 lecture XII but with $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + u\partial/\partial x + w\partial/\partial y$

$$\frac{1}{\gamma p}\frac{\mathrm{D}^{\pm}p}{\mathrm{D}t} \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}u}{\mathrm{D}t} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{\bar{t}_N} - \frac{\partial w}{\partial y} \qquad \frac{\mathrm{D}^{\pm}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} \pm (a \pm u)\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\dot{\mathrm{w}}}{\bar{t}_N} \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{q_m}{c_pT_u}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

### *Distinguished limit*

Near the CJ regime the instability threshold concerns transonic conditions associated with small heat release

Clavin Williams 2009

Same distinguished limit as in pp 9-10 lecture XII

$$\boxed{\epsilon^2 \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_pT_u} \ll 1 \qquad (\gamma-1) = O(\epsilon)}$$

⚠ notation

With the notations of p.10 lecture XII $\quad \mathrm{t} \equiv \frac{t}{\bar{t}_N}, \quad \mathrm{x} \equiv \frac{x}{a_u\bar{t}_N}, \quad \breve{u} \equiv \frac{u}{a_u}, \quad \breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{p}{p_u}\right), \quad \breve{\theta} \equiv \frac{(T-T_u)}{T_u}$ one introduces $\boxed{\breve{\nu} \equiv w/a_u \text{ and } \mathrm{y} \equiv y/a_u\bar{t}_N}$

Anticipating that the transverse convection $w\partial/\partial y$ introduces negligible corrections, the reduced equations take the form

acoustic wave
$$\left[\frac{\partial}{\partial \mathrm{t}} \pm (1 \pm \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} \pm \breve{u}) = \epsilon^2\dot{\mathrm{w}} - \frac{\partial \breve{\nu}}{\partial \mathrm{y}} \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\breve{\nu} = -\frac{\partial \breve{\pi}}{\partial \mathrm{y}}$$
vorticity wave

$\dot{\mathrm{w}}(\psi, \theta)$
$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}} \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma-1)\breve{\pi}] = \epsilon^2\dot{\mathrm{w}}$$
entropy wave

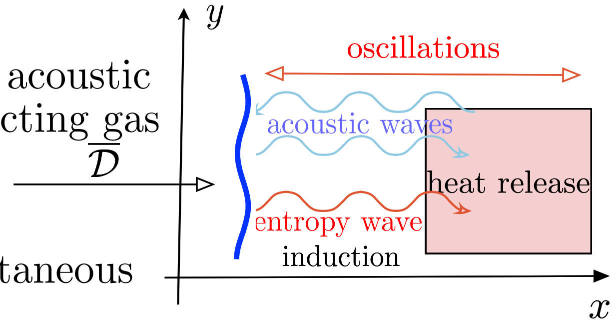
Boundary conditions: Rankine-Hugoniot at the shock front and boundedness condition in the burnt gas $\mathrm{x} \to \infty$

# Scalings

## *Time scale*

As in p. 11 lecture XII the slow time scale is controlled by the upstream-running acoustic wave in the feed back loop between the shock and the reacting gas

$$\boxed{\tau \equiv \frac{t}{\bar{t}_N/\epsilon} = \epsilon\,\mathrm{t}} \qquad \mathrm{t} \equiv t/\bar{t}_N \qquad \tau = O(1) \qquad \partial/\partial\mathrm{t} = \epsilon\partial/\partial\tau$$




the downstream propagating acoustic wave and the voracity wave are quasi instantaneous

## *Longitudinal variations*

$q_m \ll c_p T_u \quad \Rightarrow$ the variation across the detonation thickness are small

$$\breve{u} \equiv \frac{u}{a_u} = 1 + \epsilon\mu, \quad \breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{p}{p_u}\right) = \epsilon\pi, \quad \breve{\theta} \equiv \frac{(T - T_u)}{T_u} = \epsilon^2\theta \qquad \boxed{\mu = O(1), \quad \pi = O(1), \quad \theta = O(1)}$$

## *Transverse scaling* (obtained by the linear approximation of the Rankine-Hugoniot relations)

Rankine-Hugoniot
$w_N = (\mathcal{D} - u_N)\alpha'_y$
(p.5 lecture IV, p.6 lecture XIII)

$$\Rightarrow \quad \xi = 0: \quad \breve{\nu} = 2\epsilon\sqrt{f}\partial \mathrm{a}/\partial\mathrm{y}, \quad \partial\breve{\nu}/\partial\mathrm{y} = 2\epsilon\sqrt{f}\partial^2\mathrm{a}/\partial\mathrm{y}^2 \quad \text{where} \quad \mathrm{x} = \mathrm{a}(\epsilon\mathrm{t}, \cancel{\mathrm{y}})$$

$\mathrm{y}\sqrt{\epsilon}$

non-dimensional equation of the wrinkled shock front
$\mathrm{a} = \text{amplitude}/(a_u\bar{t}_N)$

$$\left[\frac{\partial}{\partial\mathrm{t}} \pm (1 \pm \breve{u})\frac{\partial}{\partial\mathrm{x}}\right](\breve{\pi} \pm \breve{u}) = \epsilon^2\dot{\mathrm{w}} - \frac{\partial\breve{\nu}}{\partial\mathrm{y}} \Rightarrow \partial\breve{\nu}/\partial\mathrm{y} = O(\epsilon^2) \Rightarrow \partial^2\mathrm{a}/\partial\mathrm{y}^2 = O(\epsilon) \Rightarrow \begin{cases} \mathrm{y} \equiv y/(a_u\bar{t}_N) = O(1/\sqrt{\epsilon}) \\ \breve{\nu} \equiv w/a_u = O(\epsilon^{3/2}) \end{cases}$$

$$\boxed{\eta \equiv \mathrm{y}\sqrt{\epsilon} = O(1) \qquad \breve{\nu} = \epsilon^{3/2}\nu \qquad \nu = O(1)} \quad \Rightarrow \quad \boxed{\mathrm{x} = \mathrm{a}(\tau, \eta)}$$

$\dot{\mathrm{a}}_\tau \equiv \partial\mathrm{a}/\partial\tau = O(1)$

$\mathrm{a}'_\eta \equiv \partial\mathrm{a}/\partial\eta = O(1)$

## *Leading order relations*

transverse convection $w\dfrac{\partial}{\partial y} = \dfrac{\epsilon^2}{\bar{t}_N}\nu\dfrac{\partial}{\partial\eta}$ is negligible in front of the unsteady term $\dfrac{\partial}{\partial t} = \dfrac{\epsilon}{\bar{t}_N}\dfrac{\partial}{\partial\tau}$

downstream propagating acoustic wave

$$\left[\frac{\partial}{\partial\mathrm{t}} + (1 + \breve{u})\frac{\partial}{\partial\mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2\dot{\mathrm{w}} - \frac{\partial\breve{\nu}}{\partial\mathrm{y}} \quad \Rightarrow \quad \boxed{\frac{\partial}{\partial\mathrm{x}}(\pi + \mu) = 0}$$

same relations as in the planar case
see p.11 lecture XII

entropy-vorticity wave

$$\left[\frac{\partial}{\partial\mathrm{t}} + \breve{u}\frac{\partial}{\partial\mathrm{x}}\right][\breve{\theta} - (\gamma - 1)\breve{\pi}] = \epsilon^2\dot{\mathrm{w}} \qquad (\gamma - 1) \equiv \epsilon h \qquad \boxed{\frac{\partial}{\partial\xi}[\theta - h\pi - \psi] \approx 0}$$

$$\left[\frac{\partial}{\partial\mathrm{t}} + \breve{u}\frac{\partial}{\partial\mathrm{x}}\right]\psi = \dot{\mathrm{w}} \qquad \left[\frac{\partial}{\partial\mathrm{t}} + \breve{u}\frac{\partial}{\partial\mathrm{x}}\right]\breve{\nu} = -\frac{\partial\breve{\pi}}{\partial\mathrm{y}} \quad \Rightarrow \quad \boxed{\frac{\partial\nu}{\partial\mathrm{x}} = -\frac{\partial\pi}{\partial\eta}}$$

additional relation in the transverse direction
(vorticity wave)

# Model for CJ or near CJ regimes

Clavin Williams 2009, 2012

In the moving frame $x = a(\eta, \tau)$

$$\tau = \epsilon t, \quad \eta = \sqrt{\epsilon} y, \quad \xi \equiv x - a(\eta,\tau), \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} \to \sqrt{\epsilon}\left(\frac{\partial}{\partial \eta} - a'_\eta \frac{\partial}{\partial \xi}\right), \quad \frac{\partial}{\partial t} \to \epsilon\left(\frac{\partial}{\partial \tau} - \dot{a}_\tau \frac{\partial}{\partial \xi}\right)$$

the equations for the downstream running acoustic mode and the entropy-vorticity wave yield

$$\frac{\partial}{\partial \xi}(\pi + \mu) = 0 \qquad \frac{\partial}{\partial \xi}[\theta - h\pi - \psi] \approx 0 \qquad \boxed{\frac{\partial \psi}{\partial \xi} = \dot{w}(\theta, \psi)} \qquad \frac{\partial \nu}{\partial \xi} \approx -\frac{\partial \pi}{\partial \eta} + a'_\eta \frac{\partial \pi}{\partial \xi}$$

The boundary conditions at $\xi = 0$ (Neumann state) for $\pi$ and $\theta$ are given by the Rankine-Hugoniot conditions in p.7 of lecture X where $M_u$ is replaced by $\dfrac{(\mathcal{D} - \partial\alpha/\partial t)}{a_u\,[1 + (\partial\alpha/\partial y)^2]^{1/2}}$ that is, to leading order, $M_u \to 1 + \epsilon\left[\sqrt{f} - \dot{a}_\tau - (1/2)(a'_\eta)^2\right] + ..$

the first nonlinear correction is purely geometrical

Up to first order, the boundary conditions at $\xi = 0$ for $\theta$, $\mu$ and $\pi$ are the same as in the planar case p 12 lecture XII where $\dot{a}_\tau \to \dot{a}_\tau + (1/2)(a'_\eta)^2$

$$\boxed{\xi = 0:} \quad \mu + \pi = \sqrt{f}, \quad \boxed{\mu = -\sqrt{f} + 2[\dot{a}_\tau + (1/2)(a'_\eta)^2]}, \qquad \theta = 2h[\sqrt{f} - \dot{a}_\tau - (1/2)(a'_\eta)^2] \quad \boxed{\psi = 0}$$

$$\forall\, \xi \geqslant 0: \quad \pi = -\mu + \sqrt{f}, \quad \boxed{\theta = h\sqrt{f} - h\mu + \psi}$$

same relations as in the planar case see p. 13 lecture XII

## *Upstream-running acoustic wave*

additional terms coming from the front wrinkling

$$\left[\frac{\partial}{\partial t} - (1 - \breve{u})\frac{\partial}{\partial x}\right](\breve{\pi} - \breve{u}) = \epsilon^2 \dot{w} - \frac{\partial \breve{v}}{\partial y} \quad \Rightarrow \quad \boxed{2\left[\frac{\partial}{\partial \tau} + (\mu - \dot{a}_\tau)\frac{\partial}{\partial \xi}\right]\mu = -\dot{w}(\theta, \psi) + \frac{\partial \nu}{\partial \eta} - a'_\eta \frac{\partial \nu}{\partial \xi}}$$

where $\nu$ is solution to $\boxed{\dfrac{\partial \nu}{\partial \xi} = \dfrac{\partial \mu}{\partial \eta} - a'_\eta \dfrac{\partial \mu}{\partial \xi}}$ with the boundary condition $\boxed{\xi = 0: \quad \nu = 2\sqrt{f}a'_\eta - 2[\dot{a}_\tau + (1/2)(a'_\eta)^2]a'_\eta}$

$x = \alpha: \quad w = (\mathcal{D} - u)\alpha'_y$ (p.5 lecture IV)

**3 first order PDEs for $\nu$, $\mu$ and $\psi$ with 3 boundary conditions at $\xi = 0$**

**An integral equation for $a(\eta, \tau)$ is obtained when applying the downstream boundary condition**

$$\xi \to \infty: \quad \psi = 1, \quad \dot{w} = 0, \quad \mu = \overline{\mu}_b = -\sqrt{f - 1}$$

see Clavin-Williams 2009 for a more general condition: radiation condition

# Multidimensional stability analysis (analytical expressions)

Analytical expressions for the linear growth rate vs the wave number, written $\sigma(\kappa)$ in non-dimensional form, can be obtained for a simplified reaction rate, assuming that it depends on temperature only at the Neumann state

$$\dot{\mathrm{w}}(\theta,\psi) \approx \dot{\mathrm{w}}(\theta_N,\psi) \qquad \text{with} \qquad (E/k_B\overline{T}_N)(\overline{T}_b-\overline{T}_u)/\overline{T}_b = O(1)$$

This approximation is well verified for the main mechanism of instability that is associated with the variation of the induction length

## *Model equation*

Then the linear problem is reduced to solve a single ODE of second order (with variable coefficients)

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} - \sigma\frac{\mathrm{d}Y}{\mathrm{d}\zeta} - \frac{\kappa^2}{2}|\overline{\mu}|Y = \frac{1}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} + \frac{\sigma}{2}\mathrm{h}|\overline{\mu}|\Omega'_N$$

where $\mathrm{d}\zeta = \mathrm{d}\xi/|\overline{\mu}(\xi)|$, $\Omega(\xi)$ is the distribution of heat release rate in the steady state and $\Omega'_N(\xi)$ is the distribution denoting the thermal sensitivity ( see p.8 lecture XII)

The dispersion relation is obtained by applying the 3 boundary conditions:

$$\zeta = 0: \ Y = -2\sqrt{f}, \quad \mathrm{d}Y/\mathrm{d}\zeta = -2\sigma\sqrt{f}, \qquad \zeta\to\infty: \ Y = 0$$

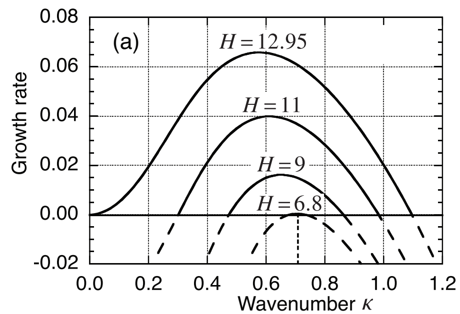
Clavin Williams 2009

## *Analytical result*

The equation for $\sigma$ becomes polynomial for a particular example $\Omega(\xi) = \dfrac{\xi^n}{n!}\mathrm{e}^{-\xi}$, $\Omega'_N(\xi) = \dfrac{\mathrm{d}(\xi\Omega)}{\mathrm{d}\xi}$ and $|\overline{\mu}| \approx 1$

$$\boxed{4\left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)^{n+2} = H\sigma + \left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)}$$

single parameter
$H \equiv (n+1)\beta_N(\gamma-1)$

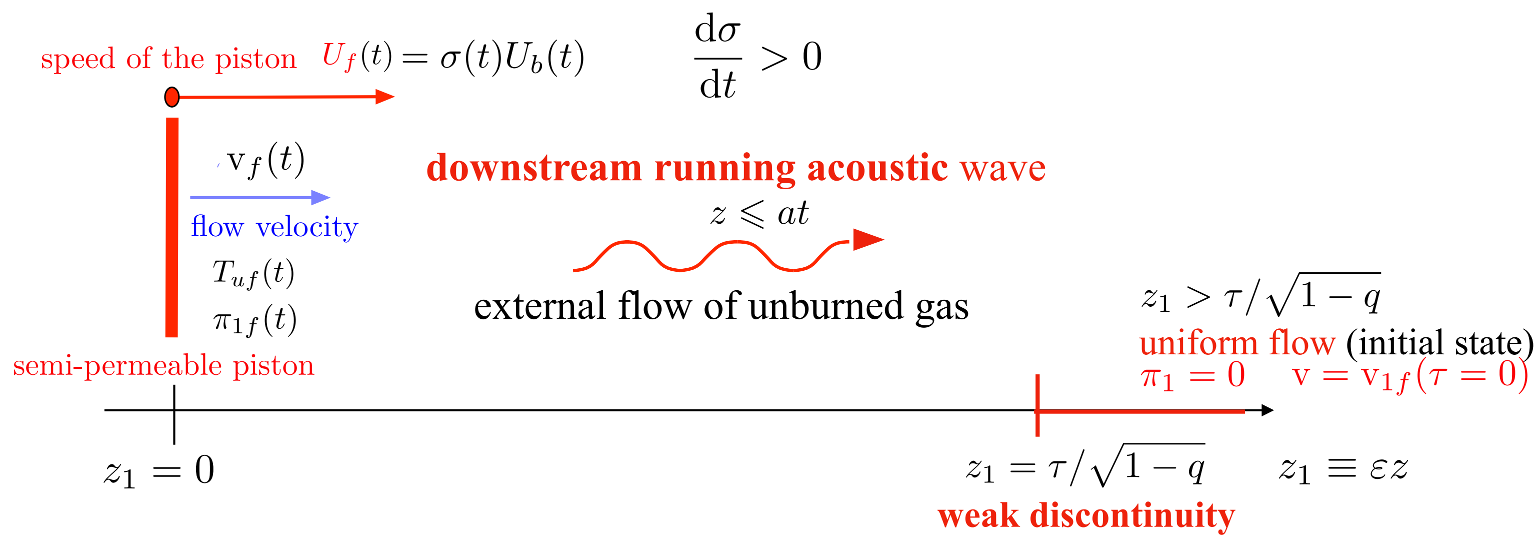



The multidimensional instability develops at a finite wave length (larger than the detonation thickness by a factor $(M_u^2-1)^{-1/2}$) when increasing the thermal sensitivity $\beta_N$ or the induction length $n$. The Poincaré-Andronov (Hopf) bifurcation occurs before the planar instability with a pulsating frequency larger than the transit time by a factor $(M_u^2-1)^{-1}$

Bifurcation scenario similar to that of the strongly overdriven regimes

see the scaling of length and time p.11

# Compressible external flow upstream from the flame




**acoustic wave**
$$\frac{\partial^2 \pi_1}{\partial \tau^2} = \frac{1}{(1-q)} \frac{\partial^2 \pi_1}{\partial z_1^2} \qquad \frac{\partial v}{\partial \tau} = -\frac{1}{\gamma} \frac{\partial \pi_1}{\partial z_1}$$

$$z = 0: \quad \pi_1 = \pi_{1f}(\tau) \qquad v = v_f(\tau)$$

$$z \geqslant \frac{\tau}{\sqrt{1-q}}: \quad \pi_1 = \pi_{1f}(\tau = 0) = 0 \qquad v = v_f(\tau = 0)$$

$$\tau = 0: \quad \pi_1 = \pi_{1f}(\tau = 0) = 0 \qquad v = v_f(\tau = 0)$$

**downstream running** acoustic wave
$$\pi_1 = \pi_{1f}(\tau - \sqrt{1-q}\, z_1) \qquad v = v_f(\tau - \sqrt{1-q}\, z_1) \longrightarrow \frac{\partial v}{\partial z_1} = \frac{\sqrt{(1-q)}}{\gamma} \frac{\partial \pi_1}{\partial z_1}$$

**Riemann** solution (linear approximation) **before the shock formation** on the leading edge

spatial integration

$$v_f(\tau) = v_f(\tau = 0) + \frac{\sqrt{1-q}}{\gamma} \pi_{1f}(\tau)$$

**pressure** versus **flow** velocity **on the flame**

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