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Structure and Dynamics of Combustion Waves in Premixed Gases

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Lecture XV
Cellular detonations

Lecture 15 : Cellular detonations

15-1. Cellular detonations at strong overdrive

Order of magnitude. Scaling

Formulation

Outer flow in the burnt gas

Inner structure

Matching

Linear growth rate

Weakly nonlinear analysis

15-2. Cellular instability near the CJ condition

Formulation

Scaling

Model for CJ or near CJ regimes

Multidimensional stability analysis

XV-1) Cellular detonations at strong overdrive

Clavin et al (1997) Clavin Denet (2002) Daou Clavin (2003)

Order of magnitude and scaling

Same limit as in § XII-3

Overdriven detonations in the Newtonian approximation

$\overline{M}_N = \overline{u}_N/\overline{a}_N$

$$\epsilon^2 \equiv \overline{M}_N^2 \ll 1, \qquad \overline{M}_u^2 = O(1/\epsilon^2), \qquad (\gamma - 1) = O(\epsilon^2)$$

$q_N \equiv q_m/c_p\overline{T}_N$ $\quad q_N = O(1) \Leftrightarrow$ Always unstable to transverse disturbances

Clavin et al (1997)

Underlying linear mechanisms see p4 lecture XIV

longitudinal oscillation of the complex shock reaction zone

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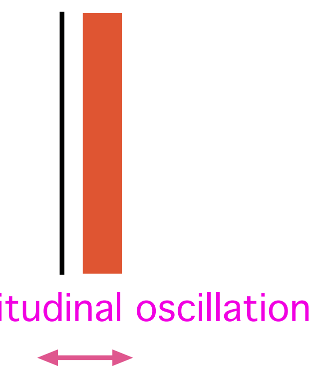
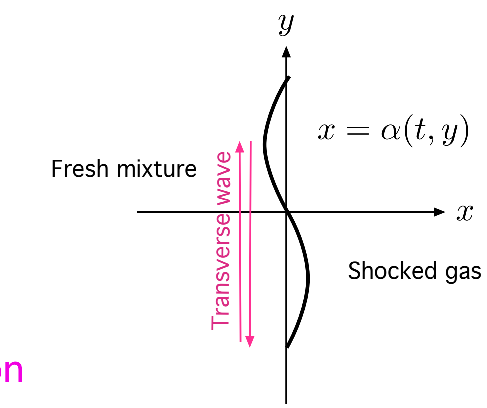
transverse oscillatory mode of the lead shock





see pp 4-8 lecture XII

period of oscillation: $\overline{t}_N \equiv \tau_r(\overline{T}_N)$

transverse velocity of the shock disturbances: $\overline{a}_N$

see pp 7-8 lecture XIV

$\Rightarrow$ wavelength of unstable disturbance $\overline{a}_N\overline{t}_N = d_N/\epsilon$

detonation thickness $d_N = \overline{u}_N\overline{t}_N$ (see p 11 lecture X)

$\overline{t}_N \equiv \tau_r(\overline{T}_N)$

The unstable wavelengths are much larger than the detonation thickness

Reduced mass-weighted coordinates of order unity (generalization of p 4 lecture XII)

$$\mathrm{x} \equiv \frac{1}{\rho_u\overline{\mathcal{D}}\overline{t}_N}\int_\alpha^x \rho(x', y, z, t)\mathrm{d}x' \qquad \mathrm{t} \equiv \frac{t}{\overline{t}_N} \qquad \mathbf{y} \equiv (\epsilon\, y/d_N, \epsilon\, z/d_N)$$

$x = \alpha(y, z, t)$ instantaneous shock position

Non-dimensional variables of order unity

denoting $\hat{w}$ the original dimensional quantities and w the dimensionless quantity

$$u \equiv \hat{u}/\overline{u}_N,\ \mathbf{v} \equiv \epsilon\hat{\mathbf{v}}/\overline{u}_N,\ p \equiv \hat{p}/\overline{p}_N,\ T \equiv \hat{T}/\overline{T}_N \ \text{ and } \alpha \equiv \hat{\alpha}/d_N,\ d_N \equiv \overline{u}_N\overline{t}_N$$

notations

where the scaling of the transverse velocity $\hat{\mathbf{v}}$ comes from the Rankine-Hugoniot condition

$\hat{\mathbf{v}}_N/\overline{u}_N \propto (\partial\hat{\alpha}/\partial y,\ \partial\hat{\alpha}/\partial z)$ and the scaling of the transverse coordinates $\partial/\partial y = \epsilon d_N^{-1}\partial/\partial \mathrm{y},\ \partial/\partial z = \epsilon d_N^{-1}\partial/\partial \mathrm{z}$

Formulation (Clavin et al. 1997, Clavin 2002)

$$\mathrm{x} \equiv \frac{1}{\overline{\rho}_N\overline{u}_N\overline{t}_N}\int_{\alpha(\mathrm{y,z,t})}^{x}\rho(x',y,z,t)\mathrm{d}x'$$

$$\mathbf{y} \equiv (\epsilon\, y/d_N, \epsilon\, z/d_N) \qquad \mathrm{t} \equiv \frac{t}{\overline{t}_N}$$

$\dfrac{m(t)}{\sqrt{1+|\nabla\alpha|^2}}$ mass flux across the leading shock

$$\Rightarrow \quad \frac{\mathrm{D}}{\mathrm{Dt}} = \frac{\partial}{\partial \mathrm{t}} + [\mathrm{m(t)} - \upsilon(\mathrm{x},\mathbf{y},\mathrm{t})]\frac{\partial}{\partial \mathrm{x}} + \mathbf{v}.\nabla_{\mathbf{y}}$$

where

$$\mathrm{m(t)} \equiv 1 - (\partial\hat{\alpha}/\partial\hat{t})/\overline{\mathcal{D}} \quad \text{and} \quad \upsilon(\mathrm{x},\mathbf{y},\mathrm{t}) \equiv \int_0^{\mathrm{x}} \nabla_{\mathbf{y}}.\mathbf{v}\,\mathrm{d}x' = O(1)$$

change of variable

$\mathbf{v}(\mathrm{t},\mathbf{y}) \qquad \nabla_{\mathbf{y}}.\mathbf{v} = O(1)$

continuity (in the linear approximation)

$$\Rightarrow \quad \frac{\partial r}{\partial \mathrm{t}} + \mathrm{m(t)}\frac{\partial r}{\partial \mathrm{x}} = \frac{\partial}{\partial \mathrm{x}}[u + \overline{r}(\mathrm{x})\upsilon(\mathrm{x},\mathbf{y},\mathrm{t})] \quad \text{where} \quad r(\mathrm{x},\mathbf{y},\mathrm{t}) \equiv \overline{\rho}_N/\hat{\rho}$$

notations

Stability limit

$q_N \equiv q_m/c_p\overline{T}_N$

$$q_N = O(\epsilon^2)$$

$$\overline{u}_N/\overline{\mathcal{D}} = O(\epsilon^2) \ \Rightarrow \ \mathrm{m(t)} = 1 + O(\epsilon^2)$$

Clavin Denet (2002) Daou Clavin (2003)

small variations across the shocked gas

Expansion in powers of ϵ^2

$$q_N = \epsilon^2 q_2 \qquad u = 1 + \epsilon^2\overline{u}_2(\mathrm{x}) + \delta u \qquad T = 1 + \epsilon^2\overline{T}_2(\mathrm{x}) + \delta T \qquad p = 1 + \epsilon^4\overline{p}_4 + \delta p$$

Linear equations

$$\epsilon^2\left[\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u - \upsilon\frac{\mathrm{d}\overline{u}}{\mathrm{dx}}\right] = -\frac{\partial \delta p}{\partial \mathrm{x}} \qquad \left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)(\nabla.\mathbf{v}) = -\overline{u}\nabla^2\delta p$$

$\overline{\rho}_N\overline{u}_N^2/\overline{p}_N = \epsilon^2$

$$\frac{1}{\gamma}\frac{\overline{u}}{\overline{p}}\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta p + \frac{\partial}{\partial \mathrm{x}}(\delta u + \overline{u}\upsilon) = q_N(\delta\dot{\mathrm{w}} + \upsilon\dot{\overline{\mathrm{w}}}), \qquad q_N = \epsilon^2 q_2$$

Rankine-Hugoniot conditions

$\dot{\alpha}_{\mathrm{t}} \equiv (\overline{t}_N/d_N)\partial\hat{\alpha}/\partial t = \overline{u}_N^{-1}\partial\hat{\alpha}/\partial t = O(1) \qquad \nabla\alpha \equiv \epsilon d_N^{-1}\left(\dfrac{\partial\hat{\alpha}}{\partial y}, \dfrac{\partial\hat{\alpha}}{\partial z}\right) = O(1)$

$$\mathrm{x} = 0: \quad \delta u \approx \left[1 + \frac{1}{M_u^2} - \frac{(\gamma-1)}{2}\right]\dot{\alpha}_{\mathrm{t}}, \quad \mathbf{v} \approx \left[1 - \frac{1}{M_u^2}\right]\nabla\alpha, \qquad \delta p \approx -2\epsilon^2\dot{\alpha}_{\mathrm{t}}, \quad \delta T_N \approx -(\gamma-1)\dot{\alpha}_{\mathrm{t}}$$

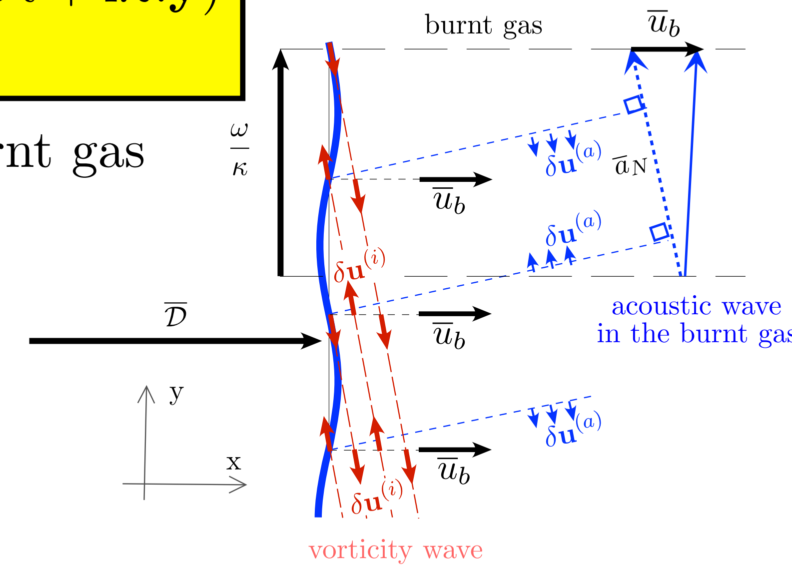
Outer flow in the burnt gas

$$\alpha = \tilde{\alpha}\exp(\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y}) \qquad \delta f(\mathrm{x},\mathbf{y},\mathrm{t}) = \tilde{f}(\mathrm{x})\tilde{\alpha}\exp(\sigma \mathrm{t} + \mathrm{i}\boldsymbol{\kappa}.\mathbf{y})$$

$$\sigma \equiv \hat{\sigma}\bar{t}_N, \qquad \kappa \equiv |\hat{k}|\overline{u}_N\bar{t}_N/\epsilon, \qquad \sigma(\kappa) = s(\kappa) \pm \mathrm{i}\omega(\kappa)$$

The flow of the unperturbed solution is uniform in the burnt gas

$$\sigma = \sigma_0 + \epsilon^2\sigma_2 + \dots$$



Acoustic waves

$$\frac{\mathrm{D}^2}{\mathrm{D}t^2}\delta p - \overline{a}_N^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta p = 0 \;\Rightarrow\; \left\{\frac{1}{\gamma}\frac{\overline{u}_b}{\overline{p}_b}\left(\frac{\mathrm{d}}{\mathrm{dx}} + \sigma\right)^2 - \frac{1}{\epsilon^2}\frac{\mathrm{d}^2}{\mathrm{dx}^2} + \overline{u}_b^2\kappa^2\right\}\tilde{p}(\mathrm{x}) = 0$$

$$\tilde{p}(\mathrm{x}) = \tilde{p}_b\tilde{\alpha}e^{il\mathrm{x}} \;\Rightarrow\; il = \frac{\varepsilon^2\sigma \pm \varepsilon\sqrt{\sigma^2 + \varkappa^2 - \varepsilon^2\varkappa^2}}{1-\varepsilon^2}$$

$\varkappa^2 \equiv (\gamma\overline{u}_b\overline{p}_b)\kappa^2 \approx \kappa^2 + \epsilon^2(h+q_2)\kappa^2 + ..$

$\varepsilon^2 \equiv (\overline{u}_b/\gamma\overline{p}_b)\epsilon^2 \approx \epsilon^2$

where $q_N \equiv \epsilon^2 q_2$ $\quad (\gamma-1) \equiv \epsilon^2 h$

anticipating

$$q_N = O(\epsilon^2) \;\Rightarrow\; \sigma^2 + \kappa^2 = O(\epsilon^2) \;\Rightarrow\; il = O(\varepsilon^2), \quad il = \epsilon^2 il_2, \quad il_2 = O(1)$$

the sound waves propagate in the burnt gas in a direction quasi-parallel to the front

modification of δp across the reaction zone is of order ϵ^4

$$\mathrm{x} = 0: \quad \delta p = -2\epsilon^2\dot{\alpha}_{\mathrm{t}} \quad \Rightarrow \quad \tilde{p}(\epsilon^2\mathrm{x}) = -2\epsilon^2\sigma\exp(\mathrm{i}\epsilon^2 l_2\mathrm{x}) + O(\epsilon^4)$$

Vorticity wave

$$\left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\delta u^{(i)} = 0, \quad \left(\frac{\partial}{\partial \mathrm{t}} + \frac{\partial}{\partial \mathrm{x}}\right)\mathbf{v}^{(i)} = 0$$

valid up to ϵ^2 in the burnt gas

Order unity

$$\mathrm{x} = 0: \; \delta u \approx \dot{\alpha}_{\mathrm{t}}, \; \mathbf{v} \approx \nabla\alpha \;\Rightarrow\; \delta u_0^{(i)} = \frac{\partial\alpha}{\partial \mathrm{t}}(\mathrm{t}-\mathrm{x},\mathbf{y}), \quad \mathbf{v}_0^{(i)} = \nabla\alpha(\mathrm{t}-\mathrm{x},\mathbf{y})$$

Continuity

$$\partial\delta u_0^{(i)}/\partial\mathrm{x} + \nabla.\mathbf{v}_0^{(i)} = 0 \;\Rightarrow\; \partial^2\alpha/\partial\mathrm{t}^2 - \nabla^2\alpha = 0, \quad \sigma_0 = \pm\mathrm{i}\kappa$$

the growth or damping rate is small of order ϵ^2

σ_2?

$$il_2 \approx \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\,\kappa^2}$$

where $q_N \equiv \epsilon^2 q_2$ $\;(\gamma-1) \equiv \epsilon^2 h$

$\alpha(\mathbf{y},\mathrm{t}) = \tilde{\alpha}e^{\sigma t + i\boldsymbol{\kappa}.\mathbf{y}}$ $\delta u = \tilde{u}(\mathrm{x})\tilde{\alpha}e^{\sigma t + i\boldsymbol{\kappa}.\mathbf{y}}$ $\delta \mathbf{v} = \tilde{\mathbf{v}}(\mathrm{x})\tilde{\alpha}e^{\sigma t + i\boldsymbol{\kappa}.\mathbf{y}}$ $\delta p = \tilde{p}(\mathrm{x})\tilde{\alpha}e^{\sigma t + i\boldsymbol{\kappa}.\mathbf{y}}$

Outer flow (burnt gas)

$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\delta u^{(i)} = 0,\quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\mathbf{v}^{(i)} = 0$ valid up to ϵ^2 in the burnt gas $\sigma_0 = \pm i\kappa$ $\sigma_2?$

$$\tilde{u}^{(i)} = \left[\sigma_0 + \epsilon^2\tilde{u}^{(i)}_{b2}\right]e^{-\sigma x} \qquad \nabla.\tilde{\mathbf{v}}^{(i)} = \left[-\kappa^2 + \epsilon^2\nabla.\tilde{\mathbf{v}}^{(i)}_{b2}\right]e^{-\sigma x}$$

$\nabla.\tilde{\mathbf{v}}^{(i)}_{b2} = \sigma_0\tilde{u}^{(i)}_{b2}$

unknown constants of integration

$\tilde{p}(\epsilon^2 x) = -2\epsilon^2\sigma e^{i\epsilon^2 l_2 x}$ $\Rightarrow$ $\tilde{u}^{(a)} = 2\epsilon^2 i l_2 e^{i\epsilon^2 l_2 x}$ $\nabla.\tilde{\mathbf{v}}^{(a)} = -2\epsilon^2\kappa^2 e^{i\epsilon^2 l_2 x}$

the acoustic flow is of order ϵ^2

$$\tilde{p}(\epsilon^2 x) = -2\epsilon^2\sigma\exp(i\epsilon^2 l_2 x) + O(\epsilon^4) \qquad il_2 \approx \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\kappa^2}$$

the acoustic flow is small, of order ϵ^2, and varies on a long length scale

Inner detonation structure (inner zone)

Inner flow (reacting gas)

splitting $\tilde{u} \equiv \tilde{U}^{(i)}(\mathrm{x}) + \tilde{u}^{(a)}(\epsilon^2 \mathrm{x}) \qquad \tilde{\mathbf{v}} \equiv \tilde{\mathbf{V}}^{(i)}(\mathrm{x}) + \tilde{\mathbf{v}}^{(a)}(\epsilon^2 \mathrm{x})$ $\tilde{U}^{(i)}(\mathrm{x}) = O(1)$ $\tilde{\mathbf{V}}^{(i)}(\mathrm{x}) = O(1)$

$$\frac{1}{\gamma}\frac{\overline{u}}{\overline{p}}\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\delta p + \frac{\partial}{\partial x}(\delta u + \overline{u}v) = q_N(\delta\dot{w} + v\overline{\dot{w}}), \Rightarrow \frac{d}{dx}\left[\tilde{U}^{(i)} + \overline{u}(x)\tilde{v}^{(i)}(x)\right] \approx q_N\left(\tilde{\dot{w}} + \tilde{v}^{(i)}_0\overline{\dot{w}}\right) \qquad \tilde{v}^{(i)} \equiv \int_0^x \nabla.\tilde{\mathbf{V}}^{(i)}dx'$$

subtracting out the acoustics

valid up to ϵ^2 $q_N = \epsilon^2 q_2$

$$d\overline{u}/dx = q_N\overline{\dot{w}} \Rightarrow d\tilde{U}^{(i)}/dx + \overline{u}\nabla.\tilde{\mathbf{V}}^{(i)} \approx q_N\tilde{\dot{w}}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)(\nabla.\mathbf{v}) = -\overline{u}\nabla^2\delta p \Rightarrow (\partial/\partial t + \partial/\partial x)\nabla.\mathbf{V}^{(i)} \approx 0$$ valid up to ϵ^2

subtracting out the acoustics

$x = 0:\quad \mathbf{v} \approx \left[1 - \frac{1}{M_u^2}\right]\nabla\alpha \qquad \nabla.\tilde{\mathbf{v}}^{(a)} = -2\epsilon^2\kappa^2 \Rightarrow \nabla.\tilde{\mathbf{V}}^{(i)} \approx \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\kappa^2 e^{-\sigma x}$ valid up to ϵ^2

Rankine-Hugoniot see p.6 lecture X and p.6 lecture XIII

$$x = 0:\quad \delta u \approx \left[1 + \frac{1}{M_u^2} - \frac{(\gamma - 1)}{2}\right]\dot{\alpha}_t \qquad \tilde{u}^{(a)} = 2\epsilon^2 i l_2 \Rightarrow \tilde{U}^{(i)}(x) - \left[1 + \frac{1}{M_U^2} - \frac{\gamma - 1}{2}\right]\sigma + 2\epsilon^2 i l_2 + \overline{u}(x)\int_0^x \nabla.\tilde{\mathbf{V}}^{(i)}dx' \approx q_N\int_0^x\left(\tilde{\dot{w}} + \tilde{v}^{(i)}_0\overline{\dot{w}}\right)dx'$$

$\sigma_0 = \pm i\kappa$ $\quad$ $\sigma = \pm i\kappa + \epsilon^2\sigma_2^2$ $\quad$ $\sigma_2?$

Matching

internal solution

$$\tilde{U}^{(i)}(\mathrm{x}) - \left[1 + \frac{1}{M_U^2} - \frac{\gamma-1}{2}\right]\sigma + 2\epsilon^2 \mathrm{i}l_2 + \overline{u}(\mathrm{x})\int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}' \approx q_N \int_0^{\mathrm{x}} \left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' \qquad q_N = \epsilon^2 q_2$$

$$\nabla.\tilde{\mathbf{V}}^{(i)} \approx \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\kappa^2 \mathrm{e}^{-\sigma\mathrm{x}} \quad \Rightarrow \quad \int_0^{\mathrm{x}} \nabla.\tilde{\mathbf{V}}^{(i)}\mathrm{dx}' = \left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\frac{\kappa^2}{\sigma}\left(1 - \mathrm{e}^{-\sigma\mathrm{x}}\right)$$

at the end of the reaction $\overline{\dot{w}} = 0,\ \tilde{\dot{w}} = 0 : \tilde{U}^{(i)}(\mathrm{x}) \rightarrow$ constant term + oscillatory term

constant term

$$\left[1 + \frac{1}{M_u^2} - \frac{\gamma-1}{2}\right]\sigma - 2\epsilon^2 \mathrm{i}l_2 - \overline{u}_b\left[-1 + \epsilon^2\left(2 + \frac{1}{\epsilon^2 M_u^2}\right)\right]\frac{\kappa^2}{\sigma} + q_N \int_0^{\infty}\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' = 0$$

$$\mathrm{i}l_2 \approx \sigma_0 - \sqrt{2\sigma_0\sigma_2 + (h + q_2 - 1)\,\kappa^2}$$

external solution

$$\tilde{u}^{(i)} = \left[\sigma_0 + \epsilon^2\tilde{u}_{b2}^{(i)}\right]\mathrm{e}^{-\sigma\mathrm{x}}$$

oscillatory term with an amplitude varying on a long length scale, $\mathrm{Re}(\sigma) = O(\epsilon^2)$

matching $\Rightarrow$ the constant term of the internal solution should be zero $\Rightarrow$ equation for σ when $\tilde{\dot{\mathrm{w}}}$ is known

Reaction rate and dispersion relation $\sigma_2(\kappa)$

$$\frac{\partial T}{\partial \mathrm{t}} + \frac{\partial T}{\partial \mathrm{x}} = q_N(1 + v_0^{(i)})\dot{\mathrm{w}} \qquad \frac{\partial \psi}{\partial \mathrm{t}} + \frac{\partial \psi}{\partial \mathrm{x}} = (1 + v_0^{(i)})\dot{\mathrm{w}} \qquad v_0^{(i)} = -\frac{\kappa^2}{\sigma_0}\left(1 - \mathrm{e}^{-\sigma_0\mathrm{x}}\right) \qquad \mathrm{x} = 0: \quad T = T_N(\mathbf{y}, t), \quad \psi = 0$$

additional effect of wrinkling

method similar to that used for galloping detonations see p.7-8 lecture XII

Clavin et al (1997)
Daou Clavin (2003)

$$\int_0^{\infty}\left(\tilde{\dot{\mathrm{w}}} + \tilde{v}_0^{(i)}\overline{\dot{\mathrm{w}}}\right)\mathrm{dx}' = \frac{\kappa^2}{\sigma}\left[-1 + \mathcal{S}^{(i)}(\kappa)\right] \quad \Rightarrow \quad \frac{\sigma_0}{\kappa}\sqrt{2\frac{\sigma_0\sigma_2}{\kappa^2} + h + q_2 - 1} - \frac{\sigma_0\sigma_2}{\kappa^2} + 1 - \frac{3}{4}h = \frac{q_2}{2}\mathcal{S}^{(i)}(\kappa)$$

equation for $\sigma_2(\kappa)$

$$\mathcal{S}^{(i)}(\kappa) \equiv \beta_N(\gamma-1)s_{\beta_N}^{(i)}(\kappa) + s_q^{(i)}(\kappa) \qquad s_{\beta_N}^{(i)}(\kappa) \equiv \int_0^{\infty}\Omega_N'(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx} \qquad s_q^{(i)}(\kappa) \equiv \int_0^{\infty}(1 + \mathrm{i}\kappa\mathrm{x})\overline{\Omega}(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

Linear growth rate

Daou Clavin (2003)

$$2\frac{\mathrm{Re}(\sigma)/\kappa,}{q_N} = -\mathrm{Im}\left[\mathcal{S}^{(i)}(\kappa)\right] - \frac{\overline{M}_N}{\sqrt{q_N}}\mathcal{S}^{(a)}(\kappa)$$

quasi-isobaric instability mechanism

$\mathrm{Im}\ \mathcal{S}^{(i)} < 0$

$$\mathcal{S}^{(i)}(\kappa) \equiv \beta_N(\gamma-1)s^{(i)}_{\beta_N}(\kappa) + s^{(i)}_q(\kappa)$$

sensitivity to T_N

$$s^{(i)}_{\beta_N}(\kappa) \equiv \int_0^\infty \Omega'_N(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

strong instability due to wrinkling

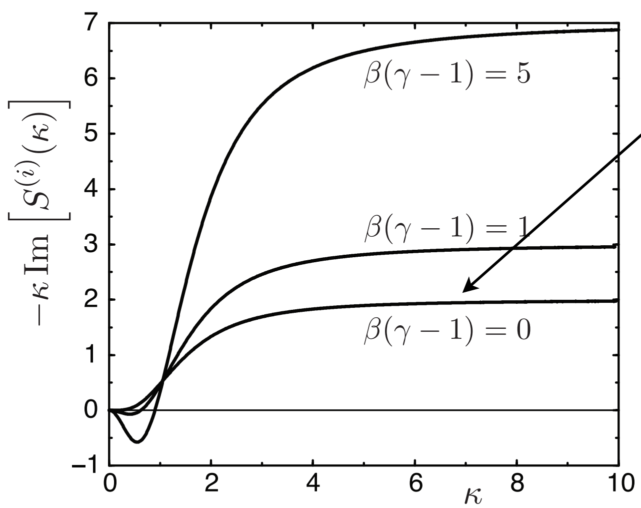
$$s^{(i)}_q(\kappa) \equiv \int_0^\infty (1+\mathrm{i}\kappa\mathrm{x})\overline{\Omega}(\mathrm{x})\mathrm{e}^{-\mathrm{i}\kappa\mathrm{x}}\mathrm{dx}$$

still working when $\beta_N = 0$

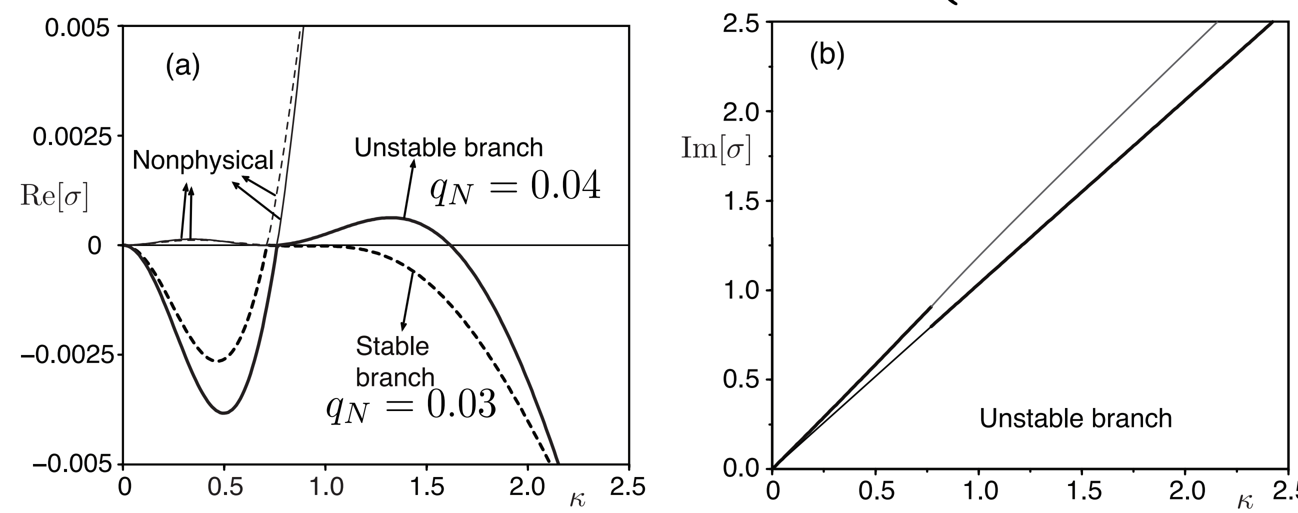
stabilizing effect due to compressibility

$$\mathcal{S}^{(a)}(\kappa) \equiv 2\left|\mathrm{Im}\sqrt{\frac{(\gamma-1)}{2q_N} + \mathcal{S}^{(i)}(\kappa) - 1}\right| > 0$$

$q_N \nearrow \Rightarrow \begin{cases} \text{stabilisation} \searrow \\ \text{instability} \nearrow \end{cases}$



Arrhenius law with $\beta_N q_N = 0.1$

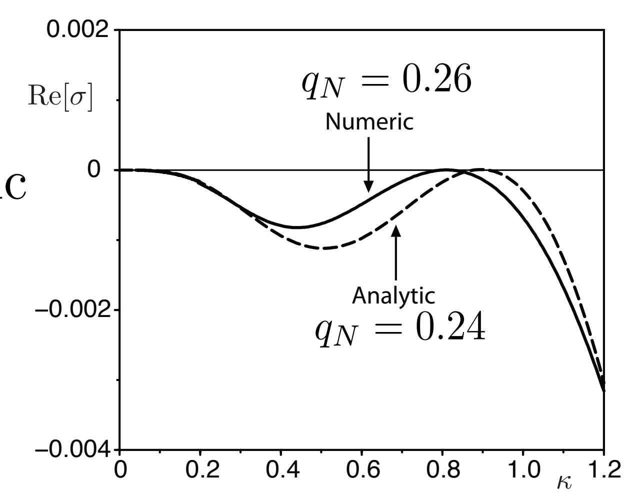


$(\gamma-1) = 0.1,\ \beta_N = 10,\ M_u^2 = 50$

Threshold of linear instability for $\beta_N = 0\ \ \gamma = 1.05,\ M_u^2 = 20$

$(M_N = 0.267)$

good agreement between theory and numeric



Weakly nonlinear analysis of cellular detonations

Clavin Denet (2002)

Near to the instability threshold the dominant nonlinear effects are those responsible for singularity formation on the inert shock front (representative of Mach stem), see p.12 of lecture XIV

Model equation

A weakly nonlinear analysis leads to a combination of the linear equation for the multidimensional instability of an overdriven detonation and the nonlinear equation for the lead shock

equation of the detonation front $x = \alpha(\mathbf{y}, \mathrm{t})$

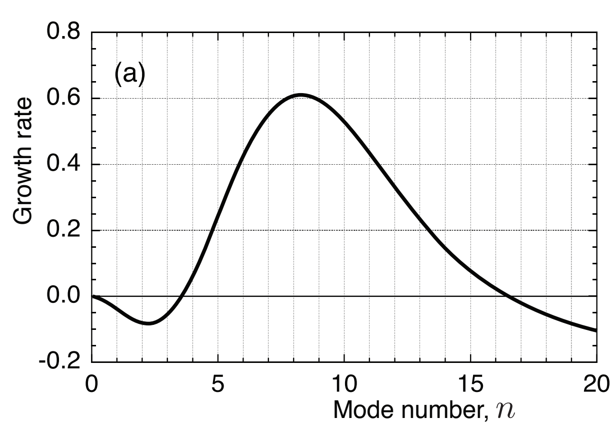
$$\frac{\partial^2 \alpha}{\partial \mathrm{t}^2} - c^2 \nabla^2 \alpha + \frac{\partial |\nabla \alpha|^2}{\partial \mathrm{t}} = q_N L^{(i)}(\alpha) - 2\overline{M}_N \sqrt{q_N} \frac{\partial}{\partial \mathrm{t}} L^{(a)}(\alpha)$$

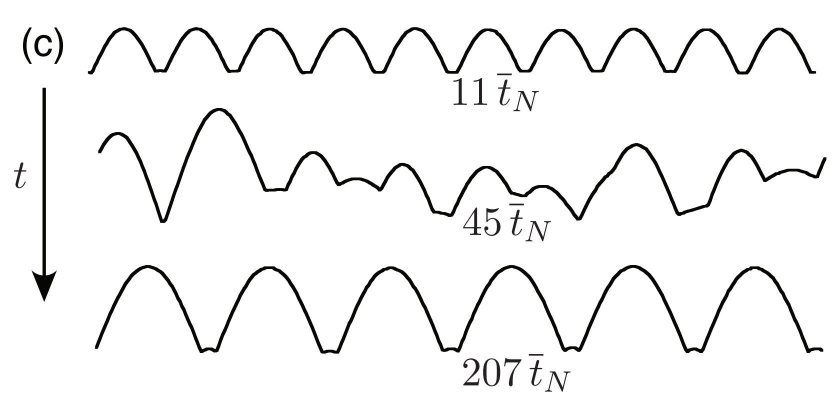
nonlinear dynamics of the lead shock — quasi-isobaric instability — stabilisation due to compressibility

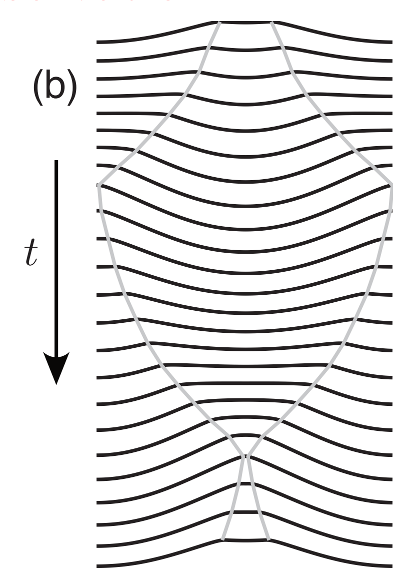
$c^2 = 1 + 3(\gamma - 1)/2$ $\quad L^{(i)}(\phi) = \beta_N(\gamma - 1) l^{(i)}_{\beta_N}(\alpha) + l^{(i)}_q(\alpha)$ $\quad L^{(a)}(\alpha) \approx \kappa\tilde{\alpha}/2$ in Fourier space

$$l^{(i)}_{\beta_N}(\alpha) = \frac{\partial^2}{\partial \mathrm{t}^2} \int_0^\infty \Omega'_N(\mathrm{x}) \alpha(\mathrm{t} - \mathrm{x}) \mathrm{dx}, \quad l^{(i)}_q(\alpha) = \nabla^2 \int_0^\infty \Psi(\mathrm{x}) \alpha(\mathrm{t} - \mathrm{x})/\mathrm{dx} \quad \text{where } \Psi(\mathrm{x}) \equiv \overline{\Omega}(\mathrm{x}) + \mathrm{d}(\mathrm{x}\overline{\Omega})/\mathrm{dx}$$

Good qualitative agreement with the experimental observation







XV-2 Cellular instability near the CJ condition
(small heat release)

Clavin Williams 2009, 2012

Formulation

Extension of the analysis of galloping detonations (planar case) pp 9-13 lecture XII

Reactive Euler equations in 2-D geometry

Same as in p.9 lecture XII but with $\mathrm{D}/\mathrm{D}t \equiv \partial/\partial t + u\partial/\partial x + w\partial/\partial y$

$$\frac{1}{\gamma p}\frac{\mathrm{D}^{\pm}p}{\mathrm{D}t} \pm \frac{1}{a}\frac{\mathrm{D}^{\pm}u}{\mathrm{D}t} = \frac{q_m}{c_pT}\frac{\dot{\mathrm{w}}}{\bar{t}_N} - \frac{\partial w}{\partial y} \qquad \frac{\mathrm{D}^{\pm}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} \pm (a \pm u)\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} \qquad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\mathrm{D}\psi}{\mathrm{D}t} = \frac{\dot{\mathrm{w}}}{\bar{t}_N} \qquad \frac{1}{T}\frac{\mathrm{D}T}{\mathrm{D}t} - \frac{(\gamma-1)}{\gamma}\frac{1}{p}\frac{\mathrm{D}p}{\mathrm{D}t} = \frac{q_m}{c_pT_u}\frac{\dot{\mathrm{w}}}{\bar{t}_N}$$

Distinguished limit

Near the CJ regime the instability threshold concerns transonic conditions associated with small heat release

Clavin Williams 2009

Same distinguished limit as in pp 9-10 lecture XII

notation

$$\epsilon^2 \equiv \frac{(\gamma+1)}{2}\frac{q_m}{c_pT_u} \ll 1 \qquad (\gamma-1) = O(\epsilon)$$

With the notations of p.10 lecture XII $\mathrm{t} \equiv \frac{t}{\bar{t}_N}$, $\mathrm{x} \equiv \frac{x}{a_u\bar{t}_N}$, $\breve{u} \equiv \frac{u}{a_u}$, $\breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{p}{p_u}\right)$, $\breve{\theta} \equiv \frac{(T-T_u)}{T_u}$ one introduces $\breve{\nu} \equiv w/a_u$ and $\mathrm{y} \equiv y/a_u\bar{t}_N$

Anticipating that the transverse convection $w\partial/\partial y$ introduces negligible corrections, the reduced equations take the form

acoustic wave
$$\left[\frac{\partial}{\partial \mathrm{t}} \pm (1 \pm \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} \pm \breve{u}) = \epsilon^2\dot{\mathrm{w}} - \frac{\partial \breve{\nu}}{\partial \mathrm{y}}$$

vorticity wave
$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\breve{\nu} = -\frac{\partial \breve{\pi}}{\partial \mathrm{y}}$$

$\dot{\mathrm{w}}(\psi,\,\theta)$
$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}}$$

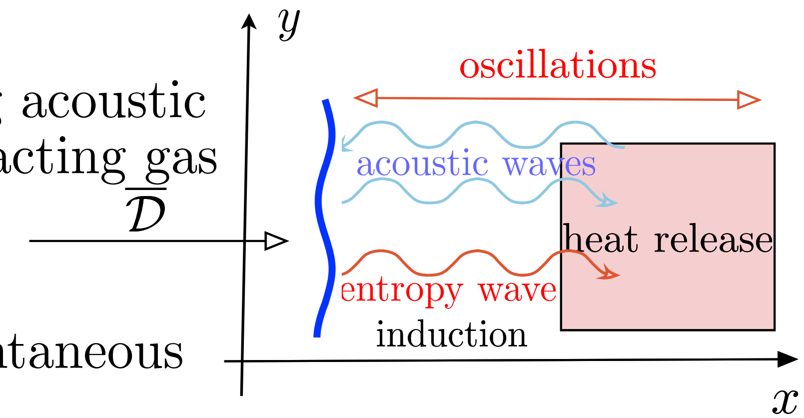
entropy wave
$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma-1)\breve{\pi}] = \epsilon^2\dot{\mathrm{w}}$$

Boundary conditions: Rankine-Hugoniot at the shock front and boundedness condition in the burnt gas $\mathrm{x} \to \infty$

Scalings

Time scale

As in p. 11 lecture XII the slow time scale is controlled by the upstream-running acoustic wave in the feed back loop between the shock and the reacting gas



$$\tau \equiv \frac{t}{\bar{t}_N/\epsilon} = \epsilon\,\mathrm{t} \qquad \mathrm{t} \equiv t/\bar{t}_N \qquad \tau = O(1) \qquad \partial/\partial \mathrm{t} = \epsilon \partial/\partial\tau$$

the downstream propagating acoustic wave and the voracity wave are quasi instantaneous

Longitudinal variations

$q_m \ll c_p T_u \quad \Rightarrow$ the variation across the detonation thickness are small

$$\breve{u} \equiv \frac{u}{a_u} = 1 + \epsilon\mu, \qquad \breve{\pi} \equiv \frac{1}{\gamma}\ln\left(\frac{p}{p_u}\right) = \epsilon\pi, \qquad \breve{\theta} \equiv \frac{(T - T_u)}{T_u} = \epsilon^2\theta$$

$$\mu = O(1), \qquad \pi = O(1), \qquad \theta = O(1)$$

Transverse scaling (obtained by the linear approximation of the Rankine-Hugoniot relations)

Rankine-Hugoniot
$w_N = (\mathcal{D} - u_N)\alpha'_y$
(p.5 lecture IV, p.6 lecture XIII)

$$\Rightarrow \quad \xi = 0: \quad \breve{\nu} = 2\epsilon\sqrt{f}\,\partial \mathrm{a}/\partial \mathrm{y}, \quad \partial\breve{\nu}/\partial \mathrm{y} = 2\epsilon\sqrt{f}\,\partial^2 \mathrm{a}/\partial \mathrm{y}^2$$

where $\mathrm{x} = \mathrm{a}(\epsilon \mathrm{t}, \not{\mathrm{y}})$ $\mathrm{y}\sqrt{\epsilon}$

non-dimensional equation of the wrinkled shock front
$\mathrm{a} = \text{amplitude}/(a_u \bar{t}_N)$

$$\left[\frac{\partial}{\partial \mathrm{t}} \pm (1 \pm \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} \pm \breve{u}) = \epsilon^2 \dot{\mathrm{w}} - \frac{\partial \breve{\nu}}{\partial \mathrm{y}} \Rightarrow \partial\breve{\nu}/\partial \mathrm{y} = O(\epsilon^2) \Rightarrow \partial^2 \mathrm{a}/\partial \mathrm{y}^2 = O(\epsilon) \Rightarrow \begin{cases} \mathrm{y} \equiv y/(a_u \bar{t}_N) = O(1/\sqrt{\epsilon}) \\ \breve{\nu} \equiv w/a_u = O(\epsilon^{3/2}) \end{cases}$$

$$\eta \equiv \mathrm{y}\sqrt{\epsilon} = O(1) \qquad \breve{\nu} = \epsilon^{3/2}\nu \qquad \nu = O(1) \qquad \Rightarrow \qquad \mathrm{x} = \mathrm{a}(\tau, \eta)$$

$\dot{\mathrm{a}}_\tau \equiv \partial \mathrm{a}/\partial\tau = O(1)$

$\mathrm{a}'_\eta \equiv \partial \mathrm{a}/\partial\eta = O(1)$

Leading order relations

transverse convection $w\dfrac{\partial}{\partial y} = \dfrac{\epsilon^2}{\bar{t}_N}\nu\dfrac{\partial}{\partial\eta}$ is negligible in front of the unsteady term $\dfrac{\partial}{\partial t} = \dfrac{\epsilon}{\bar{t}_N}\dfrac{\partial}{\partial\tau}$

downstream propagating acoustic wave

$$\left[\frac{\partial}{\partial \mathrm{t}} + (1 + \breve{u})\frac{\partial}{\partial \mathrm{x}}\right](\breve{\pi} + \breve{u}) = \epsilon^2 \dot{\mathrm{w}} - \frac{\partial\breve{\nu}}{\partial \mathrm{y}} \quad \Rightarrow \quad \frac{\partial}{\partial \mathrm{x}}(\pi + \mu) = 0$$

same relations as in the planar case
see p.11 lecture XII

entropy-vorticity wave

$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right][\breve{\theta} - (\gamma - 1)\breve{\pi}] = \epsilon^2 \dot{\mathrm{w}} \qquad (\gamma - 1) \equiv \epsilon h \qquad \frac{\partial}{\partial\xi}[\theta - h\pi - \psi] \approx 0$$

$$\left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\psi = \dot{\mathrm{w}} \qquad \left[\frac{\partial}{\partial \mathrm{t}} + \breve{u}\frac{\partial}{\partial \mathrm{x}}\right]\breve{\nu} = -\frac{\partial\breve{\pi}}{\partial \mathrm{y}} \quad \Rightarrow \quad \frac{\partial\nu}{\partial \mathrm{x}} = -\frac{\partial\pi}{\partial\eta}$$

additional relation in the transverse direction
(vorticity wave)

Model for CJ or near CJ regimes

Clavin Williams 2009, 2012

In the moving frame $x = a(\eta, \tau)$

$$\tau = \epsilon t, \quad \eta = \sqrt{\epsilon} y, \quad \xi \equiv x - a(\eta,\tau), \quad \frac{\partial}{\partial x} \to \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} \to \sqrt{\epsilon}\left(\frac{\partial}{\partial \eta} - a'_\eta \frac{\partial}{\partial \xi}\right), \quad \frac{\partial}{\partial t} \to \epsilon \left(\frac{\partial}{\partial \tau} - \dot{a}_\tau \frac{\partial}{\partial \xi}\right)$$

the equations for the downstream running acoustic mode and the entropy-vorticity wave yield

$$\frac{\partial}{\partial \xi}(\pi + \mu) = 0 \qquad \frac{\partial}{\partial \xi}[\theta - h\pi - \psi] \approx 0 \qquad \boxed{\frac{\partial \psi}{\partial \xi} = \dot{w}(\theta, \psi)} \qquad \frac{\partial \nu}{\partial \xi} \approx -\frac{\partial \pi}{\partial \eta} + a'_\eta \frac{\partial \pi}{\partial \xi}$$

The boundary conditions at $\xi = 0$ (Neumann state) for π and θ are given by the Rankine-Hugoniot conditions in p.7 of lecture X where M_u is replaced by $\dfrac{(\mathcal{D} - \partial\alpha/\partial t)}{a_u\,[1 + (\partial\alpha/\partial y)^2]^{1/2}}$ that is, to leading order, $M_u \to 1 + \epsilon\left[\sqrt{f} - \dot{a}_\tau - (1/2)(a'_\eta)^2\right] + ..$

the first nonlinear correction is purely geometrical

Up to first order, the boundary conditions at $\xi = 0$ for θ, μ and π are the same as in the planar case p 12 lecture XII where $\dot{a}_\tau \to \dot{a}_\tau + (1/2)(a'_\eta)^2$

$$\boxed{\xi = 0 :} \quad \mu + \pi = \sqrt{f}, \quad \boxed{\mu = -\sqrt{f} + 2[\dot{a}_\tau + (1/2)(a'_\eta)^2]}, \qquad \theta = 2h[\sqrt{f} - \dot{a}_\tau - (1/2)(a'_\eta)^2] \quad \boxed{\psi = 0}$$

$$\forall\, \xi \geqslant 0 : \quad \pi = -\mu + \sqrt{f}, \quad \boxed{\theta = h\sqrt{f} - h\mu + \psi}$$ same relations as in the planar case see p. 13 lecture XII

Upstream-running acoustic wave

additional terms coming from the front wrinkling

$$\left[\frac{\partial}{\partial t} - (1 - \breve{u})\frac{\partial}{\partial x}\right](\breve{\pi} - \breve{u}) = \epsilon^2 \dot{w} - \frac{\partial \breve{\nu}}{\partial y} \quad \Rightarrow \quad \boxed{2\left[\frac{\partial}{\partial \tau} + (\mu - \dot{a}_\tau)\frac{\partial}{\partial \xi}\right]\mu = -\dot{w}(\theta, \psi) + \frac{\partial \nu}{\partial \eta} - a'_\eta \frac{\partial \nu}{\partial \xi}}$$

where ν is solution to $\boxed{\dfrac{\partial \nu}{\partial \xi} = \dfrac{\partial \mu}{\partial \eta} - a'_\eta \dfrac{\partial \mu}{\partial \xi}}$ with the boundary condition $\boxed{\xi = 0 : \quad \nu = 2\sqrt{f}a'_\eta - 2[\dot{a}_\tau + (1/2)(a'_\eta)^2]a'_\eta}$

$x = \alpha : \quad w = (\mathcal{D} - u)\alpha'_y$ (p.5 lecture IV)

3 first order PDEs for ν, μ and ψ with 3 boundary conditions at $\xi = 0$

An integral equation for $a(\eta, \tau)$ is obtained when applying the downstream boundary condition

$$\xi \to \infty : \quad \psi = 1, \quad \dot{w} = 0, \quad \mu = \overline{\mu}_b = -\sqrt{f - 1}$$

see Clavin-Williams 2009 for a more general condition: radiation condition

Multidimensional stability analysis (analytical expressions)

Analytical expressions for the linear growth rate vs the wave number, written $\sigma(\kappa)$ in non-dimensional form, can be obtained for a simplified reaction rate, assuming that it depends on temperature only at the Neumann state

$$\dot{\mathrm{w}}(\theta,\psi) \approx \dot{\mathrm{w}}(\theta_N,\psi) \qquad \text{with} \quad (\gamma-1)\beta_N = O(1) \quad \beta_N = O(1/\epsilon^2)$$ see p. 7 lecture XII

This approximation is well verified for the main mechanism of instability that is associated with the variation of the induction length

Model equation

Then the linear problem is reduced to solve a single ODE of second order (with variable coefficients)

$$\frac{\mathrm{d}^2Y}{\mathrm{d}\zeta^2} - \sigma\frac{\mathrm{d}Y}{\mathrm{d}\zeta} - \frac{\kappa^2}{2}|\overline{\mu}|Y = \frac{1}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}\zeta} + \frac{\sigma}{2}\mathrm{h}|\overline{\mu}|\Omega'_N$$

where $\mathrm{d}\zeta = \mathrm{d}\xi/|\overline{\mu}(\xi)|$, $\Omega(\xi)$ is the distribution of heat release rate in the steady state and $\Omega'_N(\xi)$ is the distribution denoting the thermal sensitivity (see p.8 lecture XII)

The dispersion relation is obtained by applying the 3 boundary conditions:

$$\zeta = 0: \; Y = -2\sqrt{f}, \quad \mathrm{d}Y/\mathrm{d}\zeta = -2\sigma\sqrt{f}, \qquad \zeta \to \infty: \; Y = 0$$

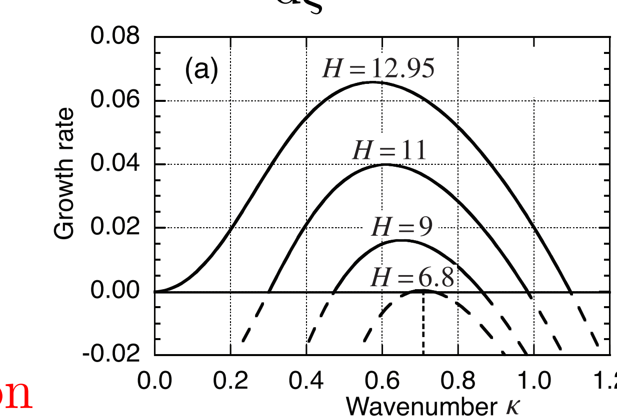
Clavin Williams 2009

Analytical result

The equation for σ becomes polynomial for a particular example $\Omega(\xi) = \frac{\xi^n}{n!}\mathrm{e}^{-\xi}$, $\Omega'_N(\xi) = \frac{\mathrm{d}(\xi\Omega)}{\mathrm{d}\xi}$ and $|\overline{\mu}| \approx 1$

$$4\left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)^{n+2} = H\sigma + \left(1+\frac{\sigma+\sqrt{\sigma^2+2\kappa^2}}{2}\right)$$

single parameter $H \equiv (n+1)\beta_N(\gamma-1)$



The multidimensional instability develops at a finite wave length (larger than the detonation thickness by a factor $(M_u^2-1)^{-1/2}$) when increasing the thermal sensitivity β_N or the induction length n. The Poincaré-Andronov (Hopf) bifurcation occurs before the planar instability with a pulsating frequency larger than the transit time by a factor $(M_u^2-1)^{-1}$

Bifurcation scenario similar to that of the strongly overdriven regimes

see the scaling of length and time p.11